ASYMPTOTIC BEHAVIOR OF THE ARROW–HURWICZ DIFFERENTIAL SYSTEM WITH TIKHONOV REGULARIZATION

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Abstract. In a real Hilbert space setting, we investigate the asymptotic behavior of the solutions of the classical Arrow–Hurwicz differential system combined with Tikhonov regularizing terms. Under some newly proposed conditions on the Tikhonov terms involved, we show that the solutions of the regularized Arrow–Hurwicz differential system strongly converge toward the element of least norm within its set of zeros. Moreover, we provide fast asymptotic decay rate estimates for the so-called "primal-dual gap function" and the norm of the solutions' velocity. If, in addition, the Tikhonov regularizing terms are decreasing, we provide some refined estimates in the sense of an exponentially weighted moving average. Under the additional assumption that the governing operator of the Arrow–Hurwicz differential system satisfies a reverse Lipschitz condition, we further provide a fast rate of strong convergence of the solutions toward the unique zero. We conclude our study by deriving the corresponding decay rate estimates with respect to the so-called "viscosity curve". Numerical experiments illustrate our theoretical findings.

1. Introduction

Let X and Y be real Hilbert spaces endowed with inner products $\langle \cdot\,, \cdot \rangle_X$, $\langle \cdot\,, \cdot \rangle_Y$ and associated norms $\|\cdot\|_X$, $\|\cdot\|_Y$. Consider the minimization problem

$$\text{(P)} \qquad \min\,\{f(x) : Ax = b\},$$

where $f : X \to \mathbb{R}$ is a convex and continuously differentiable function, $A : X \to Y$ a linear and continuous operator, and $b \in Y$. We assume that the (closed and convex) set of optimal solutions of (P) is non-empty, i.e.,

$$S := \{x \in C : f(x) = \inf_C f\} \neq \emptyset$$

with $C := \{x \in X : Ax = b\}$ denoting the feasible set of (P). Recall that (P) admits an optimal solution whenever C is non-empty and, for instance, f is coercive, that is, $\lim_{\|x\|_X \to +\infty} f(x) = +\infty$.

Let us associate with (P) the Lagrangian

$$\begin{aligned} L : X \times Y &\longrightarrow \mathbb{R} \\ (x, \lambda) &\longmapsto f(x) + \langle \lambda, Ax - b\rangle_Y \end{aligned}$$

which, by construction, is a convex-concave and continuously differentiable bifunction. Classically, the convex minimization problem (P) admits an equivalent representation in terms of the saddle-value problem

$$\min_{x \in X} \sup_{\lambda \in Y} L(x, \lambda).$$

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It is well known (see, e.g., Ekeland and Témam [19]) that $\bar{x} \in X$ is an optimal solution of (P), and $\bar{\lambda} \in Y$ a corresponding Lagrange multiplier, if and only if $(\bar{x}, \bar{\lambda})$ is a saddle point of L, that is,

$$L(\bar{x}, \lambda) \leq L(\bar{x}, \bar{\lambda}) \leq L(x, \bar{\lambda}) \quad \forall (x, \lambda) \in X \times Y.$$

Equivalently, $(\bar{x}, \bar{\lambda}) \in X \times Y$ is a saddle point of L if and only if $(\bar{x}, \bar{\lambda})$ satisfies the system of primal-dual optimality conditions

$$\begin{cases} \nabla f(x) + A^* \lambda = 0 \\ \qquad\quad Ax - b = 0 \end{cases}$$

with $\nabla f : X \to X$ denoting the gradient of f, and $A^* : Y \to X$ the adjoint operator of A. Throughout, we denote by $M \subset Y$ the (possibly empty, closed, and convex) set of Lagrange multipliers associated with (P). Recall that a Lagrange multiplier (and thus, a saddle point of L) exists, for example, whenever the constraint qualification

$$b \in \operatorname{sri} A(X)$$

is verified. Here, for a convex set $K \subset Y$, we denote by

$$\operatorname{sri} K = \Big\{ x \in K : \bigcup_{\mu > 0} \mu(K - x) \text{ is a closed linear subspace of } Y \Big\}$$

its strong relative interior; we refer the reader to Bauschke and Combettes [10] (see also Boţ [11]) for a detailed exposition of constraint qualifications.

In this work, we investigate the nonautonomous differential system

$$\text{(AHT)} \qquad \begin{cases} \dot{x} + \nabla f(x) + A^* \lambda + \varepsilon(t) x = 0 \\ \qquad\quad \dot{\lambda} + b - Ax + \varepsilon(t) \lambda = 0 \end{cases}$$

relative to the convex minimization problem (P). The (AHT) evolution system essentially combines the classical "generalized steepest descent dynamics" introduced by Arrow and Hurwicz [1] (see also Kose [24] and Arrow et al. [2]) with Tikhonov regularizing terms; cf. Tikhonov and Arsénine [30]. Here, $\varepsilon : [t_0, +\infty[\to]0, +\infty[$ denotes, for some $t_0 \geq 0$, the Tikhonov regularization function which is assumed to be continuously differentiable such that

$$\lim_{t \to +\infty} \varepsilon(t) = 0.$$

In view of this regularization, the (AHT) differential system is governed by the perturbed operator

$$T_t := T + \varepsilon(t) \operatorname{Id},$$

where

$$\begin{aligned} T : X \times Y &\longrightarrow X \times Y \\ (x, \lambda) &\longmapsto \big(\nabla_x L(x, \lambda), -\nabla_\lambda L(x, \lambda) \big) \end{aligned}$$

is the maximally monotone operator associated with the "saddle function" L; see, e.g., Rockafellar [28, 29]. Noticing that the zeros of T are nothing but the saddle points of L, i.e.,

$$\operatorname{zer} T = S \times M,$$

and the Tikhonov regularization function $\varepsilon(t)$ is vanishing as $t \to +\infty$, we may expect that the solutions $(x(t), \lambda(t))$ of (AHT) converge, as $t \to +\infty$, toward an element in $S \times M$.

As it turns out, the asymptotic behavior of the solutions of (AHT) depends critically on the rate at which $\varepsilon(t)$ tends to zero as $t \to +\infty$. In the particular case when $\varepsilon(t)$ vanishes "sufficiently fast" as $t \to +\infty$, in the sense that

$$\int_{t_0}^{\infty} \varepsilon(\tau)\,\mathrm{d}\tau < +\infty,$$

the solutions of (AHT) are known to inherit the asymptotic properties of the ones of the classical Arrow–Hurwicz differential system; see, e.g., the general results in Attouch et al. [5] (see also Cominetti et al. [17]). As such, we may only expect the weak ergodic convergence of the solutions of (AHT) toward their asymptotic center in $S \times M$; see Niederländer [25, 26] for the corresponding results on the classical Arrow–Hurwicz evolution system.

On the other hand, if the Tikhonov regularization function $\varepsilon(t)$ vanishes "slowly" as $t \to +\infty$, in the sense that

$$\int_{t_0}^{\infty} \varepsilon(\tau)\,\mathrm{d}\tau = +\infty,$$

the solutions of (AHT) are asymptotically dominated by the regularizing terms. In this case, the solutions of (AHT) are known to be strongly convergent toward the element of least norm in $S \times M$, provided that $\varepsilon(t)$ satisfies, in addition, the "finite-length property" (see Cominetti et al. [17])

$$\int_{t_0}^{\infty} |\dot{\varepsilon}(\tau)|\,\mathrm{d}\tau < +\infty,$$

or the limiting condition (see Boţ and Nguyen [12])

$$\lim_{t \to +\infty} \frac{|\dot{\varepsilon}(t)|}{\varepsilon(t)} = 0.$$

We note that the strong convergence has been previously established under the assumption that $\varepsilon(t)$ is decreasing with

$$\lim_{t \to +\infty} \frac{\dot{\varepsilon}(t)}{\varepsilon^2(t)} = 0;$$

see, e.g., Israel Jr. and Reich [22], Attouch and Cominetti [7] (see also Browder [15] and Reich [27]). Ever since, the subject of combining first- and second-order dynamics with Tikhonov regularizing terms has gained significant attention. We only mention here the recent works of Battahi et al. [9] and Boţ and Nguyen [12] in the context of first-order differential systems, and Attouch and Czarnecki [8], Attouch et al. [4, 6] for earlier studies on second-order evolution systems.

In this work, we focus on the derivation of fast convergence rates for the solutions of the (AHT) differential system. Under the assumption that $\varepsilon(t)$ is twice continuously differentiable such that there exists $t_+ \geq t_0$ with

$$\left.\begin{aligned} \varepsilon^2(t) + \dot{\varepsilon}(t) \geq 0 \\ 2\varepsilon(t)\dot{\varepsilon}(t) + \ddot{\varepsilon}(t) \leq 0 \end{aligned}\right\} \quad \forall t \geq t_+,$$

we show that the solutions $(x(t), \lambda(t))$ of (AHT) strongly converge, as $t \to +\infty$, to the element of least norm in $S \times M$, i.e.,

$$\lim_{t \to +\infty} (x(t), \lambda(t)) = \operatorname{proj}_{S \times M}(0, 0).$$

Moreover, we prove that the solutions $(x(t),\lambda(t))$ of (AHT) obey, for every $(\bar{x},\bar{\lambda})\in S\times M$, the asymptotic estimates

$$\begin{aligned}\|(\dot{x}(t),\dot{\lambda}(t))+\varepsilon(t)((x(t),\lambda(t))-(\bar{x},\bar{\lambda}))\|^2 &= \mathcal{O}\big(\,\mathrm{e}^{-2\rho(t)}+\varepsilon^2(t)\big) \text{ as } t\to+\infty;\\ \varepsilon(t)\big(L(x(t),\bar{\lambda})-L(\bar{x},\lambda(t))\big) &= \mathcal{O}\big(\,\mathrm{e}^{-2\rho(t)}+\varepsilon^2(t)\big) \text{ as } t\to+\infty;\\ \|T(x(t),\lambda(t))-T(\bar{x},\bar{\lambda})\|^2 &= \mathcal{O}\big(\,\mathrm{e}^{-2\rho(t)}+\varepsilon^2(t)\big) \text{ as } t\to+\infty;\\ \|(\dot{x}(t),\dot{\lambda}(t))\|^2 &= \mathcal{O}\big(\,\mathrm{e}^{-2\rho(t)}+\varepsilon^2(t)\big) \text{ as } t\to+\infty,\end{aligned}$$

with $\rho:[t_0,+\infty[\,\to\mathbb{R}$ being defined as

$$\rho(t)=\int_{t_0}^{t}\varepsilon(\tau)\,\mathrm{d}\tau.$$

The latter essentially recovers the decay rate estimate

$$\|(\dot{x}(t),\dot{\lambda}(t))\|^2=\mathcal{O}\big(\,\mathrm{e}^{-2\rho(t)}+|\dot{\varepsilon}(t)|\big) \text{ as } t\to+\infty$$

recently obtained by Boţ and Nguyen [12] in the context of general monotone operator flows with Tikhonov regularization. If, in addition, the Tikhonov regularization function $\varepsilon(t)$ is decreasing, we show that the following refined estimate holds:

$$\lim_{t\to+\infty}\mathrm{e}^{-\rho(t)}\int_{t_+}^{t}\mathrm{e}^{\rho(t)}\,\frac{1}{\varepsilon(\tau)}\|(\dot{x}(\tau),\dot{\lambda}(\tau))\|^2\,\mathrm{d}\tau<+\infty.$$

In the particular case $\varepsilon(t)=1/t$ with $t_0>0$, we above estimate reduces to

$$\lim_{t\to+\infty}\frac{1}{t}\int_{t_0}^{t}\tau^2\|(\dot{x}(\tau),\dot{\lambda}(\tau))\|^2\,\mathrm{d}\tau<+\infty,$$

which suggests a fast decay of the quantity $\|(\dot{x}(t),\dot{\lambda}(t))\|^2$ as $t\to+\infty$ in the sense of an exponentially weighted moving average which, in turn, induces some asymptotic "smoothing effect". Moreover, under the assumption that there exists $\alpha>0$ such that for every $(x,\lambda),(\xi,\eta)\in X\times Y$, it holds that

$$\text{(L)}\qquad \|T(x,\lambda)-T(\xi,\eta)\|^2\geq\alpha\|(x,\lambda)-(\xi,\eta)\|^2,$$

we show that the solutions $(x(t),\lambda(t))$ of (AHT) obey, for $(\bar{x},\bar{\lambda})\in S\times M$, the decay rate estimates

$$\begin{aligned}\|(x(t),\lambda(t))-(\bar{x},\bar{\lambda})\|^2 &= \mathcal{O}\big(\,\mathrm{e}^{-2\rho(t)}+\varepsilon^2(t)\big) \text{ as } t\to+\infty;\\ \|(\dot{x}(t),\dot{\lambda}(t))\|^2 &= \mathcal{O}\big(\big(1+\varepsilon^2(t)\big)\big(e^{-2\rho(t)}+\varepsilon^2(t)\big)\big) \text{ as } t\to+\infty.\end{aligned}$$

We conclude our work by deriving similar asymptotic estimates for the (AHT) solutions with respect to the so-called "viscosity curve" (x_t,λ_t) which is governed by the unique zero of the $\varepsilon(t)$-strongly monotone operator T_t, viz., for every $t\geq t_0$,

$$T(x_t,\lambda_t)+\varepsilon(t)(x_t,\lambda_t)=(0,0).$$

In particular, we show that the solutions $(x(t),\lambda(t))$ of (AHT) obey the estimates

$$\begin{aligned}\|(\dot{x}(t),\dot{\lambda}(t))+\varepsilon(t)((x(t),\lambda(t))-(x_t,\lambda_t))\|^2 &= \mathcal{O}\big(\,\mathrm{e}^{-2\rho(t)}+\varepsilon^2(t)\big) \text{ as } t\to+\infty;\\ \|T(x(t),\lambda(t))-T(x_t,\lambda_t)\|^2 &= \mathcal{O}\big(\,\mathrm{e}^{-2\rho(t)}+\varepsilon^2(t)\big) \text{ as } t\to+\infty,\end{aligned}$$

relative to the viscosity curve (x_t, λ_t). If, moreover, T verifies condition (L), we have the following refined asymptotic estimates:

$$\|T(x(t), \lambda(t)) - T(x_t, \lambda_t)\|^2 = \mathcal{O}\big(\big(1+\varepsilon^2(t)\big)\big(e^{-2\rho(t)} + \varepsilon^2(t)\big)\big) \text{ as } t \to +\infty;$$
$$\|(x(t), \lambda(t)) - (x_t, \lambda_t)\|^2 = \mathcal{O}\big(\big(1+\varepsilon^2(t)\big)\big(e^{-2\rho(t)} + \varepsilon^2(t)\big)\big) \text{ as } t \to +\infty.$$

Numerical experiments on a simple yet representative example illustrate the above theoretical findings.

2. Preliminaries on Tikhonov regularization

Let $X \times Y$ be endowed with the Hilbertian product structure $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_X + \langle \cdot, \cdot \rangle_Y$ and associated norm $\|\cdot\|$. Consider now, for each $t \geq t_0$, the regularized Lagrangian

$$\begin{aligned} L_t : X \times Y &\longrightarrow \mathbb{R} \\ (x, \lambda) &\longmapsto L(x, \lambda) + \frac{\varepsilon(t)}{2}\big(\|x\|_X^2 - \|\lambda\|_Y^2\big) \end{aligned}$$

relative to the convex minimization problem (P). Observing that L_t is $\varepsilon(t)/2$-strongly convex-concave, it follows that L_t admits, for each $t \geq t_0$, the unique saddle point $(x_t, \lambda_t) \in X \times Y$, i.e.,

$$L_t(x_t, \lambda) \leq L_t(x_t, \lambda_t) \leq L_t(x, \lambda_t) \quad \forall (x, \lambda) \in X \times Y.$$

Equivalently, the system of primal-dual optimality conditions reads

$$\begin{cases} \nabla f(x_t) + A^*\lambda_t + \varepsilon(t)x_t = 0 \\ Ax_t - b - \varepsilon(t)\lambda_t = 0. \end{cases}$$

In view of the latter, we immediately observe that, for each $t \geq t_0$, the unique zero of the $\varepsilon(t)$-strongly monotone operator

$$\begin{aligned} T_t : X \times Y &\longrightarrow X \times Y \\ (x, \lambda) &\longmapsto \big(\nabla_x L_t(x, \lambda), -\nabla_\lambda L_t(x, \lambda)\big), \end{aligned}$$

that is the "generator" of the (AHT) differential system, is precisely the saddle point of L_t, that is,

$$T_t(x_t, \lambda_t) = (0, 0) \quad \Longleftrightarrow \quad (x_t, \lambda_t) = \operatorname{argminmax}_{X \times Y} L_t.$$

Let us start our discussion with a preliminary result on the asymptotic behavior of the so-called "viscosity curve" (x_t, λ_t) as $t \to +\infty$. The result is adapted from Bruck [16, Lemma 1] (see also Attouch [3], Attouch and Cominetti [7], Cominetti et al. [17, Lemma 4]).

Lemma 2.1. *Let $S \times M$ be non-empty and let $(x_t, \lambda_t) = \operatorname{argminmax}_{X \times Y} L_t$ for each $t \geq t_0$. Then the following assertions hold:*

(i) $t \mapsto (x_t, \lambda_t)$ is bounded on $[t_0, +\infty[$ and

$$\|(x_t, \lambda_t)\| \leq \|\operatorname{proj}_{S \times M}(0, 0)\| \quad \forall t \geq t_0;$$

(ii) it holds that

$$\lim_{t \to +\infty} (x_t, \lambda_t) = \operatorname{proj}_{S \times M}(0, 0).$$

Proof. *(i)* For each $t \geq t_0$, let $(x_t, \lambda_t) = \operatorname{argminmax}_{X \times Y} L_t$ and take $(\bar{x}, \bar{\lambda}) \in S \times M$. Using that (x_t, λ_t) is a saddle point of L_t, we have

$$\begin{aligned} 0 &\geq L_t(x_t, \bar{\lambda}) - L_t(\bar{x}, \lambda_t) \\ &= L(x_t, \bar{\lambda}) - L(\bar{x}, \lambda_t) + \frac{\varepsilon(t)}{2}\big(\|(x_t, \lambda_t)\|^2 - \|(\bar{x}, \bar{\lambda})\|^2\big). \end{aligned} \tag{2.1}$$

On the other hand, $(\bar{x}, \bar{\lambda})$ is a saddle point of L so that

$$L(x_t, \bar{\lambda}) - L(\bar{x}, \lambda_t) \geq 0.$$

Combining the above inequalities and subsequently dividing by $\varepsilon(t)/2$ yields

$$\|(\bar{x}, \bar{\lambda})\|^2 \geq \|(x_t, \lambda_t)\|^2.$$

The above inequality being true for every $(\bar{x}, \bar{\lambda}) \in S \times M$, we arrive at the desired estimate.

(ii) Let $(x, \lambda) \in X \times Y$ and let $(\bar{x}, \bar{\lambda}) \in X \times Y$ be a weak sequential cluster point of $(x_t, \lambda_t)_{t \geq t_0}$, that is, there exists a sequence $t_n \to +\infty$ such that $(x_{t_n}, \lambda_{t_n}) \rightharpoonup (\bar{x}, \bar{\lambda})$ weakly in $X \times Y$ as $n \to +\infty$. Substituting t by t_n in inequality (2.1) yields

$$\begin{aligned} \frac{\varepsilon(t_n)}{2}\|(x, \lambda)\|^2 &\geq L(x_{t_n}, \lambda) - L(x, \lambda_{t_n}) + \frac{\varepsilon(t_n)}{2}\|(x_{t_n}, \lambda_{t_n})\|^2 \\ &\geq L(x_{t_n}, \lambda) - L(x, \lambda_{t_n}). \end{aligned}$$

Observing that $\varepsilon(t_n) \to 0$ as $n \to +\infty$, we obtain

$$\begin{aligned} 0 &\geq \limsup_{n \to +\infty} \big(L(x_{t_n}, \lambda) - L(x, \lambda_{t_n})\big) \\ &\geq \liminf_{n \to +\infty} L(x_{t_n}, \lambda) + \liminf_{n \to +\infty} (-L(x, \lambda_{t_n})) \\ &\geq L(\bar{x}, \lambda) - L(x, \bar{\lambda}) \end{aligned}$$

thanks to the weak lower semi-continuity of $L(\,\cdot\,, \lambda)$ and $-L(x, \,\cdot\,)$, as $L(\,\cdot\,, \lambda)$ and $-L(x, \,\cdot\,)$ are both convex and lower semi-continuous. The above inequalities being true for every $(x, \lambda) \in X \times Y$, we conclude that $(\bar{x}, \bar{\lambda})$ is a saddle point of L, that is, $(\bar{x}, \bar{\lambda}) \in S \times M$.

On the other hand, using *(i)* and owing to the weak lower semi-continuity of the norm $\|\cdot\|$, we obtain

$$\|\mathrm{proj}_{S \times M}(0, 0)\| \geq \liminf_{n \to +\infty} \|(x_{t_n}, \lambda_{t_n})\| \geq \|(\bar{x}, \bar{\lambda})\|,$$

implying that $(\bar{x}, \bar{\lambda}) = \mathrm{proj}_{S \times M}(0, 0)$. Consequently, $\mathrm{proj}_{S \times M}(0, 0)$ is the only possible weak sequential cluster point of $(x_t, \lambda_t)_{t \geq t_0}$ so that $(x_t, \lambda_t) \rightharpoonup \mathrm{proj}_{S \times M}(0, 0)$ weakly in $X \times Y$ as $t \to +\infty$. Upon relying on *(i)* again, we have

$$\begin{aligned} \|\mathrm{proj}_{S \times M}(0, 0)\| &\geq \limsup_{t \to +\infty} \|(x_t, \lambda_t)\| \\ &\geq \liminf_{t \to +\infty} \|(x_t, \lambda_t)\| \geq \|\mathrm{proj}_{S \times M}(0, 0)\| \end{aligned}$$

and thus,

$$\lim_{t \to +\infty} \|(x_t, \lambda_t)\| = \|\mathrm{proj}_{S \times M}(0, 0)\|.$$

Now, as we have both, $(x_t, \lambda_t) \rightharpoonup \mathrm{proj}_{S \times M}(0, 0)$ weakly in $X \times Y$ and $\|(x_t, \lambda_t)\| \to \|\mathrm{proj}_{S \times M}(0, 0)\|$ strongly in $X \times Y$ as $t \to +\infty$, we classically deduce

$$\lim_{t \to +\infty} (x_t, \lambda_t) = \mathrm{proj}_{S \times M}(0, 0),$$

concluding the result. $\square$

Remark 2.2. In view of the above result, we readily observe that (x_t, λ_t) obeys, for every $(\bar{x}, \bar{\lambda}) \in S \times M$, the asymptotic estimate

$$L(x_t, \bar{\lambda}) - L(\bar{x}, \lambda_t) = \mathcal{O}(\varepsilon(t)) \text{ as } t \to +\infty.$$

We show in Section 4 that a comparable estimate holds with respect to the solutions of the (AHT) differential system.

Remark 2.3. We note that the strong convergence of (x_t, λ_t) toward $\operatorname{proj}_{S\times M}(0,0)$ as $t \to +\infty$ may also be deduced from the perturbed operator

$$T_t = T + \varepsilon(t) \operatorname{Id}$$

by using the graph-closedness property of the maximally monotone operator T with respect to the weak-strong topology; see, e.g., Brézis [14, Theorem 2.2], Bauschke and Combettes [10, Theorem 23.44].

The following result, adapted from Attouch [3, Proposition 5.3] (see also Attouch and Cominetti [7], Torrabla [31, Lemma 5.2], Attouch et al. [4, Lemma 2]), provides some differential information on the viscosity curve (x_t, λ_t).

Lemma 2.4. *Let $(x_t, \lambda_t) = \operatorname{argminmax}_{X\times Y} L_t$ for each $t \geq t_0$. Then $t \mapsto (x_t, \lambda_t)$ is Lipschitz continuous on the compact intervals of $[t_0, +\infty[$ and*

$$-\dot{\varepsilon}(t)\langle (x_t, \lambda_t), (\dot{x}_t, \dot{\lambda}_t)\rangle \geq \varepsilon(t)\|(\dot{x}_t, \dot{\lambda}_t)\|^2 \quad \text{a.e. } t \geq t_0.$$

Proof. Let $(x_t, \lambda_t) = \operatorname{argminmax}_{X\times Y} L_t$ and $(x_s, \lambda_s) = \operatorname{argminmax}_{X\times Y} L_s$ for some $t > s \geq t_0$. Utilizing that L_t is $\varepsilon(t)/2$-strongly convex-concave, we have

$$\begin{aligned} 0 &\geq L_t(x_t, \lambda_s) - L_t(x_s, \lambda_t) + \frac{\varepsilon(t)}{2}\|(x_t, \lambda_t) - (x_s, \lambda_s)\|^2 \\ &= L(x_t, \lambda_s) - L(x_s, \lambda_t) + \varepsilon(t)\langle (x_t, \lambda_t), (x_t, \lambda_t) - (x_s, \lambda_s)\rangle. \end{aligned}$$

Similarly, L_s is $\varepsilon(s)/2$-strongly convex-concave so that

$$\begin{aligned} 0 &\geq L_s(x_s, \lambda_t) - L_s(x_t, \lambda_s) + \frac{\varepsilon(s)}{2}\|(x_s, \lambda_s) - (x_t, \lambda_t)\|^2 \\ &= L(x_s, \lambda_t) - L(x_t, \lambda_s) + \varepsilon(s)\langle (x_s, \lambda_s), (x_s, \lambda_s) - (x_t, \lambda_t)\rangle. \end{aligned}$$

Combining the above inequalities gives

$$0 \geq \langle \varepsilon(t)(x_t, \lambda_t) - \varepsilon(s)(x_s, \lambda_s), (x_t, \lambda_t) - (x_s, \lambda_s)\rangle.$$

Equivalently, we have

$$\begin{aligned} 0 \geq{}& (\varepsilon(t) - \varepsilon(s))\langle (x_t, \lambda_t), (x_t, \lambda_t) - (x_s, \lambda_s)\rangle \\ &+ \varepsilon(s)\|(x_t, \lambda_t) - (x_s, \lambda_s)\|^2. \end{aligned} \tag{2.2}$$

Since $\varepsilon(t)$ is continuously differentiable, it is Lipschitz continuous on the compact intervals of $[t_0, +\infty[$. In view of the above inequality, it readily follows that (x_t, λ_t) is Lipschitz continuous on the compact intervals of $[t_0, +\infty[$ as well and thus, differentiable almost everywhere. Upon dividing inequality (2.2) by $(t-s)^2$ and letting $s \to t$, for almost every $t \geq t_0$, we obtain

$$0 \geq \dot{\varepsilon}(t)\langle (x_t, \lambda_t), (\dot{x}_t, \dot{\lambda}_t)\rangle + \varepsilon(t)\|(\dot{x}_t, \dot{\lambda}_t)\|^2,$$

concluding the desired inequality. □

Remark 2.5. In view of the Cauchy–Schwarz inequality, we readily deduce from the above result that $(\dot{x}_t, \dot{\lambda}_t)$ obeys the asymptotic estimate

$$\|(\dot{x}_t, \dot{\lambda}_t)\| = \mathcal{O}\Big(\frac{|\dot{\varepsilon}(t)|}{\varepsilon(t)}\Big) \text{ as } t \to +\infty.$$

Let us next investigate the derivative of the viscosity curve under the additional assumption that $\varepsilon(t)$ is twice continuously differentiable such that

$$\lim_{t\to+\infty} \varepsilon(t) = 0.$$

The following result asserts that the quantity $\varepsilon(t)\|(\dot{x}_t, \dot{\lambda}_t)\|^2$ vanishes at a fast rate as $t \to +\infty$ in the sense of an exponentially weighted moving average.

Lemma 2.6. *Let $S \times M$ be non-empty and let $(x_t, \lambda_t) = \operatorname{argminmax}_{X\times Y} L_t$ for each $t \geq t_0$. Suppose that there exists $t_+ \geq t_0$ such that*

$$\left.\begin{aligned} \varepsilon^2(t) + \dot{\varepsilon}(t) \geq 0 \\ 2\varepsilon(t)\dot{\varepsilon}(t) + \ddot{\varepsilon}(t) \leq 0 \end{aligned}\right\} \quad \forall t \geq t_+.$$

Then, as $t \to +\infty$, it holds that

$$\mathrm{e}^{-2\rho(t)} \int_{t_+}^{t} \mathrm{e}^{2\rho(\tau)}\, \varepsilon(\tau)\|(\dot{x}_\tau, \dot{\lambda}_\tau)\|^2 \,\mathrm{d}\tau = \mathcal{O}\big(\mathrm{e}^{-2\rho(t)} + \varepsilon^2(t)\big).$$

Proof. Let $(x_t, \lambda_t) = \operatorname{argminmax}_{X\times Y} L_t$ and take $(\bar{x}, \bar{\lambda}) \in S\times M$. Let $\sigma : [t_0, +\infty[\to \mathbb{R}$ be defined by $\sigma(t) = \varepsilon^2(t) + \dot{\varepsilon}(t)$ such that $\dot{\sigma}(t) = 2\varepsilon(t)\dot{\varepsilon}(t) + \ddot{\varepsilon}(t)$. In view of the system of primal-dual optimality conditions, for almost every $t \geq t_0$, we have[1]

$$\Big\langle \frac{\mathrm{d}}{\mathrm{d}t} T(x_t, \lambda_t), (\dot{x}_t, \dot{\lambda}_t)\Big\rangle + \varepsilon(t)\|(\dot{x}_t, \dot{\lambda}_t)\|^2 + \frac{\dot{\varepsilon}(t)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(x_t, \lambda_t)\|^2 = 0.$$

Since $\varepsilon(t)$ is twice continuously differentiable, we readily obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\dot{\varepsilon}(t)\|(x_t, \lambda_t)\|^2\big) + \varepsilon(t)\big(\dot{\varepsilon}(t)\|(x_t, \lambda_t)\|^2\big) + \varepsilon(t)\|(\dot{x}_t, \dot{\lambda}_t)\|^2 \\ + \Big\langle \frac{\mathrm{d}}{\mathrm{d}t} T(x_t, \lambda_t), (\dot{x}_t, \dot{\lambda}_t)\Big\rangle - \frac{\dot{\sigma}(t)}{2}\|(x_t, \lambda_t)\|^2 = 0. \end{aligned}$$

Multiplying by $\mathrm{e}^{2\rho(t)}$ and taking into account that T is monotone gives

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{e}^{2\rho(t)}\,\dot{\varepsilon}(t)\|(x_t, \lambda_t)\|^2\big) + \mathrm{e}^{2\rho(t)}\,\varepsilon(t)\|(\dot{x}_t, \dot{\lambda}_t)\|^2 \\ - \mathrm{e}^{2\rho(t)}\,\frac{\dot{\sigma}(t)}{2}\|(x_t, \lambda_t)\|^2 \leq 0. \end{aligned}$$

Integration over $[t_+, t]$ while observing that $\dot{\sigma}(t) \leq 0$ for all $t \geq t_+$ shows that there exists $K \geq 0$ such that

$$\frac{\dot{\varepsilon}(t)}{2}\|(x_t, \lambda_t)\|^2 + \mathrm{e}^{-2\rho(t)} \int_{t_+}^{t} \mathrm{e}^{2\rho(\tau)}\, \varepsilon(\tau)\|(\dot{x}_\tau, \dot{\lambda}_\tau)\|^2 \,\mathrm{d}\tau \leq K\,\mathrm{e}^{-2\rho(t)}.$$

[1]In the following, we assume that $t \mapsto T(x_t, \lambda_t)$ is Lipschitz continuous on the compact intervals of $[t_0, +\infty[$, implying that it is differentiable almost everywhere. In Section 3, we provide conditions on T which justify this assumption.

On the other hand, multiplying inequality (2.1) by $\varepsilon(t)$ and subsequently adding it to the above inequality yields

$$\varepsilon(t)\big(L(x_t,\bar\lambda)-L(\bar x,\lambda_t)\big)+\frac{\sigma(t)}{2}\|(x_t,\lambda_t)\|^2-\frac{\varepsilon^2(t)}{2}\|(\bar x,\bar\lambda)\|^2 \\ +\mathrm{e}^{-2\rho(t)}\int_{t_+}^{t}\mathrm{e}^{2\rho(\tau)}\,\varepsilon(\tau)\|(\dot x_\tau,\dot\lambda_\tau)\|^2\,\mathrm{d}\tau\le K\,\mathrm{e}^{-2\rho(t)}\,.$$

Noticing that $(\bar x,\bar\lambda)$ is a saddle point of L and using the fact that $\sigma(t)\ge 0$ for all $t\ge t_+$, we obtain

$$\mathrm{e}^{-2\rho(t)}\int_{t_+}^{t}\mathrm{e}^{2\rho(\tau)}\,\varepsilon(\tau)\|(\dot x_\tau,\dot\lambda_\tau)\|^2\,\mathrm{d}\tau\le K\,\mathrm{e}^{-2\rho(t)}+\frac{\varepsilon^2(t)}{2}\|(\bar x,\bar\lambda)\|^2.$$

Successively dividing by $\mathrm{e}^{-2\rho(t)}+\varepsilon^2(t)$ and passing to the upper limit as $t\to+\infty$ then gives the desired estimate. □

Let us conclude this section with asymptotic decay rate estimates on the viscosity curve (x_t,λ_t) and its derivative given the additional assumption that there exists $\alpha>0$ such that for every $(x,\lambda),(\xi,\eta)\in X\times Y$, it holds that

$$\text{(L)}\qquad \|T(x,\lambda)-T(\xi,\eta)\|^2\ge\alpha\|(x,\lambda)-(\xi,\eta)\|^2.$$

Condition (L) may be interpreted as a particular instance of an error-bound condition (cf. Bolte et al. [13]) which clearly implies that T admits a unique zero.

Lemma 2.7. *Let $S\times M$ be non-empty, let $(x_t,\lambda_t)=\operatorname{argminmax}_{X\times Y}L_t$ for each $t\ge t_0$, and suppose that $T:X\times Y\to X\times Y$ satisfies condition* (L). *Then, for $(\bar x,\bar\lambda)\in S\times M$, it holds that*

$$\|(x_t,\lambda_t)-(\bar x,\bar\lambda)\|^2=\mathcal{O}\big(\varepsilon^2(t)\big)\ \text{as } t\to+\infty;$$
$$\|(\dot x_t,\dot\lambda_t)\|^2=\mathcal{O}\big(|\dot\varepsilon(t)|^2\big)\ \text{as } t\to+\infty.$$

Proof. For each $t\ge t_0$, let $(x_t,\lambda_t)=\operatorname{argminmax}_{X\times Y}L_t$ and take $(\bar x,\bar\lambda)\in S\times M$. In view of condition (L) and the system of primal-dual optimality conditions, for every $t\ge t_0$, we have

$$\varepsilon^2(t)\|(x_t,\lambda_t)\|^2=\|T(x_t,\lambda_t)-T(\bar x,\bar\lambda)\|^2 \\ \ge\alpha\|(x_t,\lambda_t)-(\bar x,\bar\lambda)\|^2.$$

Upon applying Lemma 2.1*(i)*, it follows that

$$\varepsilon^2(t)\|\mathrm{proj}_{S\times M}(0,0)\|^2\ge\alpha\|(x_t,\lambda_t)-(\bar x,\bar\lambda)\|^2.$$

Successively dividing by $\varepsilon^2(t)$ and passing to the upper limit as $t\to+\infty$ gives the desired estimate.

Consider now $(x_t,\lambda_t)=\operatorname{argminmax}_{X\times Y}L_t$ and $(x_s,\lambda_s)=\operatorname{argminmax}_{X\times Y}L_s$ for some $t>s\ge t_0$. Utilizing again condition (L), we have

$$\|T(x_t,\lambda_t)-T(x_s,\lambda_s)\|^2\ge\alpha\|(x_t,\lambda_t)-(x_s,\lambda_s)\|^2.$$

Dividing by $(t-s)^2$ and letting $s\to t$, for almost every $t\ge t_0$, we obtain

$$\Big\|\frac{\mathrm{d}}{\mathrm{d}t}T(x_t,\lambda_t)\Big\|^2\ge\alpha\|(\dot x_t,\dot\lambda_t)\|^2.$$

On the other hand, differentiating the system of primal-dual optimality conditions yields, for almost every $t \geq t_0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}T(x_t, \lambda_t) + \varepsilon(t)(\dot{x}_t, \dot{\lambda}_t) + \dot{\varepsilon}(t)(x_t, \lambda_t) = 0.$$

Consequently, we have

$$\begin{aligned} |\dot{\varepsilon}(t)|^2 \|(x_t, \lambda_t)\|^2 &= \left\| \frac{\mathrm{d}}{\mathrm{d}t}T(x_t, \lambda_t) + \varepsilon(t)(\dot{x}_t, \dot{\lambda}_t) \right\|^2 \\ &\geq \left\| \frac{\mathrm{d}}{\mathrm{d}t}T(x_t, \lambda_t) \right\|^2 + \varepsilon^2(t)\|(\dot{x}_t, \dot{\lambda}_t)\|^2 \\ &\geq \big(\alpha + \varepsilon^2(t)\big)\|(\dot{x}_t, \dot{\lambda}_t)\|^2, \end{aligned}$$

where the first inequality follows from the monotonicity of T. Upon applying again Lemma 2.1*(i)*, we obtain

$$\begin{aligned} |\dot{\varepsilon}(t)|^2 \|\mathrm{proj}_{S\times M}(0,0)\|^2 &\geq \big(\alpha + \varepsilon^2(t)\big)\|(\dot{x}_t, \dot{\lambda}_t)\|^2 \\ &\geq \alpha\|(\dot{x}_t, \dot{\lambda}_t)\|^2. \end{aligned}$$

Dividing by $|\dot{\varepsilon}(t)|^2$ and subsequently passing to the upper limit as $t \to +\infty$ concludes the result. □

Remark 2.8. We note that similar estimates can be derived under the more general assumption that the perturbed operator $T_t = T + \varepsilon(t)\,\mathrm{Id}$ is such that there exists $\alpha : [t_0, +\infty[\to]0, +\infty[$ verifying, for every $(x, \lambda), (\xi, \eta) \in X \times Y$ and $t \geq t_0$,

$$\|T_t(x, \lambda) - T_t(\xi, \eta)\|^2 \geq \alpha(t)\|(x, \lambda) - (\xi, \eta)\|^2.$$

We leave the details to the reader.

3. The (AHT) differential system

In the following, we presuppose that

(A1) $f : X \to \mathbb{R}$ is convex and continuously differentiable;
(A2) $\nabla f : X \to X$ is Lipschitz continuous on the bounded subsets of X;
(A3) $A : X \to Y$ is linear and continuous, and $b \in Y$;
(A4) $\varepsilon : [t_0, +\infty[\to]0, +\infty[$ is continuously differentiable such that

$$\lim_{t\to+\infty} \varepsilon(t) = 0.$$

Consider again the nonautonomous differential system[2]

$$\text{(AHT)} \qquad \begin{cases} \dot{x} + \nabla f(x) + A^*\lambda + \varepsilon(t)x = 0 \\ \dot{\lambda} + b - Ax + \varepsilon(t)\lambda = 0 \end{cases}$$

with initial data $(x_0, \lambda_0) \in X \times Y$. Throughout, we assume that (AHT) admits for each $(x_0, \lambda_0) \in X \times Y$ a unique (classical) solution, that is, a continuously differentiable function $(x, \lambda) : [t_0, +\infty[\to X \times Y$ which verifies (AHT) on $[t_0, +\infty[$ with $(x(t_0), \lambda(t_0)) = (x_0, \lambda_0)$ for some $t_0 \geq 0$; cf. Haraux [21, Proposition 6.2.1]. We note that the existence of the (necessarily unique) solutions of (AHT) may also be deduced from the general results on nonautonomous evolution equations governed by maximally monotone operators; see, e.g., Crandall and Pazy [18], Furuya et al. [20], and Kenmochi [23].

[2] In view of the above assumptions, we readily observe that the governing operator $T : X\times Y \to X\times Y$ of the (AHT) differential system is Lipschitz continuous on the bounded subsets of $X\times Y$.

Consider again, for each $t \geq t_0$, the regularized Lagrangian

$$\begin{aligned} L_t : X \times Y &\longrightarrow \mathbb{R} \\ (x, \lambda) &\longmapsto L(x, \lambda) + \frac{\varepsilon(t)}{2}\big(\|x\|_X^2 - \|\lambda\|_Y^2\big) \end{aligned}$$

associated with the convex minimization problem (P). In view of the $\varepsilon(t)/2$-strong convexity-concavity of the saddle function L_t, we immediately obtain that for every $(x, \lambda), (\xi, \eta) \in X \times Y$ and $t \geq t_0$, it holds that

$$\begin{aligned} \langle T_t(x, \lambda), (x, \lambda) - (\xi, \eta)\rangle &\geq L_t(x, \eta) - L_t(\xi, \lambda) \\ &\quad + \frac{\varepsilon(t)}{2}\|(x, \lambda) - (\xi, \eta)\|^2. \end{aligned} \tag{3.1}$$

Utilizing the above inequality relative to the (AHT) evolution system gives the following preliminary estimates with $\rho : [t_0, +\infty[\to \mathbb{R}$ being defined by

$$\rho(t) = \int_{t_0}^{t} \varepsilon(\tau)\,\mathrm{d}\tau.$$

Proposition 3.1. *Let $S \times M$ be non-empty and let $(x, \lambda) : [t_0, +\infty[\to X \times Y$ be a solution of* (AHT)*. Then $t \mapsto (x(t), \lambda(t))$ is bounded on $[t_0, +\infty[$. Moreover, for every $(\bar{x}, \bar{\lambda}) \in S \times M$, it holds that*

$$\lim_{t\to+\infty} \mathrm{e}^{-\rho(t)} \int_{t_0}^{t} \mathrm{e}^{\rho(\tau)} \big(L(x(\tau), \bar{\lambda}) - L(\bar{x}, \lambda(\tau))\big)\,\mathrm{d}\tau < +\infty;$$

$$\lim_{t\to+\infty} \mathrm{e}^{-\rho(t)} \int_{t_0}^{t} \mathrm{e}^{\rho(\tau)}\, \frac{\varepsilon(\tau)}{2}\|(x(\tau), \lambda(\tau))\|^2\,\mathrm{d}\tau < +\infty.$$

Proof. Let $(\bar{x}, \bar{\lambda}) \in S \times M$ and define $\phi : [t_0, +\infty[\to \mathbb{R}$ by $\phi(t) = \|(x(t), \lambda(t)) - (\bar{x}, \bar{\lambda})\|^2/2$. Taking the inner product with $(x(t), \lambda(t)) - (\bar{x}, \bar{\lambda})$ in (AHT) and subsequently applying the chain rule yields, for every $t \geq t_0$,

$$\dot{\phi}(t) + \langle T_t(x(t), \lambda(t)), (x(t), \lambda(t)) - (\bar{x}, \bar{\lambda})\rangle = 0.$$

In view of inequality (3.1), we obtain

$$\dot{\phi}(t) + \varepsilon(t)\phi(t) + L_t(x(t), \bar{\lambda}) - L_t(\bar{x}, \lambda(t)) \leq 0.$$

Equivalently, we have

$$\begin{aligned} &\dot{\phi}(t) + \varepsilon(t)\phi(t) + L(x(t), \bar{\lambda}) - L(\bar{x}, \lambda(t)) \\ &\quad + \frac{\varepsilon(t)}{2}\|(x(t), \lambda(t))\|^2 \leq \frac{\varepsilon(t)}{2}\|(\bar{x}, \bar{\lambda})\|^2. \end{aligned} \tag{3.2}$$

Multiplying the above inequality by $\mathrm{e}^{\rho(t)}$ yields

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{e}^{\rho(t)}\,\phi(t)\big) + \mathrm{e}^{\rho(t)}\,\big(L(x(t), \bar{\lambda}) - L(\bar{x}, \lambda(t))\big) \\ &\quad + \mathrm{e}^{\rho(t)}\,\frac{\varepsilon(t)}{2}\|(x(t), \lambda(t))\|^2 \leq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{\rho(t)}\|(\bar{x}, \bar{\lambda})\|^2. \end{aligned}$$

Integration over $[t_0, t]$ and subsequently dividing by $\mathrm{e}^{\rho(t)}$ entails

$$\begin{aligned} &\phi(t) + \mathrm{e}^{-\rho(t)} \int_{t_0}^{t} \mathrm{e}^{\rho(\tau)}\,\big(L(x(\tau), \bar{\lambda}) - L(\bar{x}, \lambda(\tau))\big)\,\mathrm{d}\tau \\ &\quad + \mathrm{e}^{-\rho(t)} \int_{t_0}^{t} \mathrm{e}^{\rho(\tau)}\,\frac{\varepsilon(\tau)}{2}\|(x(\tau), \lambda(\tau))\|^2\,\mathrm{d}\tau \leq \mathrm{e}^{-\rho(t)}\,\phi(t_0) + \frac{1}{2}\|(\bar{x}, \bar{\lambda})\|^2. \end{aligned} \tag{3.3}$$

Noticing that $(\bar{x}, \bar{\lambda})$ is a saddle point of L, it holds that

$$\mathrm{e}^{-\rho(t)} \int_{t_0}^{t} \mathrm{e}^{\rho(\tau)} \left(L(x(\tau), \bar{\lambda}) - L(\bar{x}, \lambda(\tau))\right) \mathrm{d}\tau \geq 0.$$

Consequently, we have

$$\phi(t) \leq \mathrm{e}^{-\rho(t)} \, \phi(t_0) + \frac{1}{2} \|(\bar{x}, \bar{\lambda})\|^2,$$

implying that $(x(t), \lambda(t))$ remains bounded on $[t_0, +\infty[$.

On the other hand, utilizing inequality (3.3) and taking into account that $\phi(t) \geq 0$, we obtain

$$\begin{aligned} &\mathrm{e}^{-\rho(t)} \int_{t_0}^{t} \mathrm{e}^{\rho(\tau)} \left(L(x(\tau), \bar{\lambda}) - L(\bar{x}, \lambda(\tau))\right) \mathrm{d}\tau \\ &+ \mathrm{e}^{-\rho(t)} \int_{t_0}^{t} \mathrm{e}^{\rho(\tau)} \, \frac{\varepsilon(\tau)}{2} \|(x(\tau), \lambda(\tau))\|^2 \, \mathrm{d}\tau \leq \mathrm{e}^{-\rho(t)} \, \phi(t_0) + \frac{1}{2} \|(\bar{x}, \bar{\lambda})\|^2. \end{aligned}$$

Passing to the limit as $t \to +\infty$ entails

$$\lim_{t \to +\infty} \mathrm{e}^{-\rho(t)} \int_{0}^{t} \mathrm{e}^{\rho(\tau)} \left(L(x(\tau), \bar{\lambda}) - L(\bar{x}, \lambda(\tau))\right) \mathrm{d}\tau < +\infty, \text{ and}$$

$$\lim_{t \to +\infty} \mathrm{e}^{-\rho(t)} \int_{t_0}^{t} \mathrm{e}^{\rho(\tau)} \, \frac{\varepsilon(\tau)}{2} \|(x(\tau), \lambda(\tau))\|^2 \, \mathrm{d}\tau < +\infty,$$

concluding the desired estimates. □

In view of the above result, the following estimate as outlined in Cominetti et al. [17] is verified whenever the Tikhonov regularization function $\varepsilon : [t_0, +\infty[\to]0, +\infty[$ is such that

$$\int_{t_0}^{\infty} \varepsilon(\tau) \, \mathrm{d}\tau = +\infty.$$

Corollary 3.2. *Under the hypotheses of Proposition 3.1, suppose that* $\varepsilon \notin \mathcal{L}^1([t_0, +\infty[)$. *Then, for every* $(\bar{x}, \bar{\lambda}) \in S \times M$, *it holds that*

$$\limsup_{t \to +\infty} \|(x(t), \lambda(t)) - (\bar{x}, \bar{\lambda})\|^2 \leq \|(\bar{x}, \bar{\lambda})\|^2 - \liminf_{t \to +\infty} \|(x(t), \lambda(t))\|^2.$$

Proof. Recall from inequality (3.2) that for every $t \geq t_0$, we have

$$\begin{aligned} &\dot{\phi}(t) + \varepsilon(t)\phi(t) + L(x(t), \bar{\lambda}) - L(\bar{x}, \lambda(t)) \\ &\quad \leq \frac{\varepsilon(t)}{2} \left(\|(\bar{x}, \bar{\lambda})\|^2 - \|(x(t), \lambda(t))\|^2\right). \end{aligned}$$

Since $(\bar{x}, \bar{\lambda})$ is a saddle point of L, it holds that

$$\dot{\phi}(t) + \varepsilon(t)\phi(t) \leq \frac{\varepsilon(t)}{2} \left(\|(\bar{x}, \bar{\lambda})\|^2 - \|(x(t), \lambda(t))\|^2\right).$$

Using that $(x(t), \lambda(t))$ remains bounded on $[t_0, +\infty[$ together with the fact that $\varepsilon \notin \mathcal{L}^1([t_0, +\infty[)$, applying Lemma A.1 entails

$$\limsup_{t \to +\infty} \phi(t) \leq \limsup_{t \to +\infty} \frac{1}{2} \left(\|(\bar{x}, \bar{\lambda})\|^2 - \|(x(t), \lambda(t))\|^2\right),$$

which is the desired estimate. □

Remark 3.3. Anchoring the above inequality to $\operatorname{proj}_{S\times M}(0,0)$ suggests that the solutions $(x(t),\lambda(t))$ of (AHT) strongly converge, as $t\to+\infty$, toward $\operatorname{proj}_{S\times M}(0,0)$ as soon as

$$\liminf_{t\to+\infty}\|(x(t),\lambda(t))\|\geq\|\operatorname{proj}_{S\times M}(0,0)\|.$$

Our next result provides sufficient conditions for this inequality to hold assuming that $\varepsilon:[t_0,+\infty[\,\to\,]0,+\infty[$ satisfies either one of the following estimates:

$$\int_{t_0}^{\infty}|\dot\varepsilon(\tau)|\,\mathrm{d}\tau<+\infty,\ \text{ or}$$

$$\int_{t_0}^{\infty}\frac{|\dot\varepsilon(\tau)|^2}{\varepsilon(\tau)}\,\mathrm{d}\tau<+\infty.$$

Theorem 3.4. *Let $S\times M$ be non-empty, let $(x,\lambda):[t_0,+\infty[\,\to X\times Y$ be a solution of* (AHT)*, and suppose that $\varepsilon\notin\mathcal{L}^1([t_0,+\infty[)$ with either $\dot\varepsilon\in\mathcal{L}^1([t_0,+\infty[)$ or $|\dot\varepsilon|^2/\varepsilon\in\mathcal{L}^1([t_0,+\infty[)$. Then, for every $(\bar x,\bar\lambda)\in S\times M$, it holds that*

$$\lim_{t\to+\infty}\big(L(x(t),\bar\lambda)-L(\bar x,\lambda(t))\big)=0;$$

$$\lim_{t\to+\infty}\|(\dot x(t),\dot\lambda(t))\|=0.$$

Proof. Let $\vartheta:[t_0,+\infty[\,\to\mathbb{R}$ be defined by $\vartheta(t)=\|(\dot x(t),\dot\lambda(t))\|^2/2$. Differentiating $\vartheta(t)$ and taking (AHT) into account yields, for almost every $t\geq t_0$,

$$\dot\vartheta(t)+\Big\langle\frac{\mathrm{d}}{\mathrm{d}t}T_t(x(t),\lambda(t)),(\dot x(t),\dot\lambda(t))\Big\rangle=0.$$

Equivalently, we have

$$\begin{aligned}\dot\vartheta(t)+2\varepsilon(t)\vartheta(t)+\Big\langle\frac{\mathrm{d}}{\mathrm{d}t}T(x(t),\lambda(t)),(\dot x(t),\dot\lambda(t))\Big\rangle&\\+\frac{\dot\varepsilon(t)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(x(t),\lambda(t))\|^2=0.&\end{aligned}\tag{3.4}$$

Let us first consider the case when $\dot\varepsilon\in\mathcal{L}^1([t_0,+\infty[)$. Multiplying the above inequality by $\mathrm{e}^{2\rho(t)}$ and using the fact that T is monotone entails

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{e}^{2\rho(t)}\,\vartheta(t)\big)+\mathrm{e}^{2\rho(t)}\,\frac{\dot\varepsilon(t)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(x(t),\lambda(t))\|^2\leq0.$$

In view of the Cauchy–Schwarz inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{e}^{2\rho(t)}\,\vartheta(t)\big)\leq\sqrt{2}\,\mathrm{e}^{\rho(t)}|\dot\varepsilon(t)|\|(x(t),\lambda(t))\|\sqrt{\mathrm{e}^{2\rho(t)}\,\vartheta(t)}.$$

Integration over $[t_+,t]$ for some fixed $t_+\geq t_0$, and owing to the fact that $(x(t),\lambda(t))$ remains bounded on $[t_0,+\infty[$, shows that there exists $K\geq0$ such that

$$\mathrm{e}^{2\rho(t)}\,\vartheta(t)\leq\mathrm{e}^{2\rho(t_+)}\,\vartheta(t_+)+\sqrt{2}K\int_{t_+}^{t}\mathrm{e}^{\rho(\tau)}|\dot\varepsilon(\tau)|\sqrt{\mathrm{e}^{2\rho(\tau)}\,\vartheta(\tau)}\,\mathrm{d}\tau.$$

Successively applying Lemma A.2 and dividing by $\mathrm{e}^{\rho(t)}$ yields

$$\begin{aligned}\sqrt{\vartheta(t)}&\leq\mathrm{e}^{-(\rho(t)-\rho(t_+))}\,\sqrt{\vartheta(t_+)}+\frac{K}{\sqrt2}\,\mathrm{e}^{-\rho(t)}\int_{t_+}^{t}\mathrm{e}^{\rho(\tau)}|\dot\varepsilon(\tau)|\,\mathrm{d}\tau\\&\leq\mathrm{e}^{-(\rho(t)-\rho(t_+))}\,\sqrt{\vartheta(t_+)}+\frac{K}{\sqrt2}\int_{t_+}^{t}|\dot\varepsilon(\tau)|\,\mathrm{d}\tau.\end{aligned}$$

Now, since we have both $\varepsilon \notin \mathcal{L}^1([t_0, +\infty[)$ and $\dot{\varepsilon} \in \mathcal{L}^1([t_0, +\infty[)$, passing to the upper limit as $t \to +\infty$ entails

$$\limsup_{t\to+\infty} \sqrt{\vartheta(t)} \leq \frac{K}{\sqrt{2}} \int_{t_+}^{\infty} |\dot{\varepsilon}(\tau)| \,\mathrm{d}\tau.$$

This inequality being true for every $t_+ \geq t_0$, letting $t_+ \to +\infty$ ensures that $\vartheta(t)$ tends to zero as $t \to +\infty$.

Let us now consider the case when $|\dot{\varepsilon}|^2/\varepsilon \in \mathcal{L}^1([t_0, +\infty[)$. Multiplying equality (3.4) by $\mathrm{e}^{\rho(t)}$ and using again the fact that T is monotone gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\,\mathrm{e}^{\rho(t)}\,\vartheta(t)\big) + \mathrm{e}^{\rho(t)}\,\varepsilon(t)\vartheta(t) + \mathrm{e}^{\rho(t)}\,\frac{\dot{\varepsilon}(t)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(x(t), \lambda(t))\|^2 \leq 0.$$

Upon applying the Cauchy–Schwarz inequality, we infer

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\,\mathrm{e}^{\rho(t)}\,\vartheta(t)\big) \leq \mathrm{e}^{\rho(t)}\,\frac{|\dot{\varepsilon}(t)|^2}{2\varepsilon(t)}\|(x(t), \lambda(t))\|^2.$$

Integration over $[t_+, t]$ for some fixed $t_+ \geq t_0$, and using again that $(x(t), \lambda(t))$ remains bounded on $[t_0, +\infty[$, shows that there exists $K \geq 0$ such that

$$\begin{aligned}
\vartheta(t) &\leq \mathrm{e}^{-(\rho(t)-\rho(t_+))}\,\vartheta(t_+) + \frac{K}{2}\,\mathrm{e}^{-\rho(t)} \int_{t_+}^{t} \mathrm{e}^{\rho(\tau)}\,\frac{|\dot{\varepsilon}(\tau)|^2}{\varepsilon(\tau)}\,\mathrm{d}\tau \\
&\leq \mathrm{e}^{-(\rho(t)-\rho(t_+))}\,\vartheta(t_+) + \frac{K}{2} \int_{t_+}^{t} \frac{|\dot{\varepsilon}(\tau)|^2}{\varepsilon(\tau)}\,\mathrm{d}\tau.
\end{aligned}$$

Observing now that $\varepsilon \notin \mathcal{L}^1([t_0, +\infty[)$ and $|\dot{\varepsilon}|^2/\varepsilon \in \mathcal{L}^1([t_0, +\infty[)$, we conclude that $\vartheta(t)$ vanishes as $t \to +\infty$.

Finally, in view of inequality (3.1) and the regularized Lagrangian L_t, for every $(\bar{x}, \bar{\lambda}) \in S \times M$ and $t \geq t_0$, we have

$$\begin{aligned}
\langle T_t(x(t), \lambda(t)), (x(t), \lambda(t)) - (\bar{x}, \bar{\lambda})\rangle + \frac{\varepsilon(t)}{2}\|(\bar{x}, \bar{\lambda})\|^2 \\
\geq L(x(t), \bar{\lambda}) - L(\bar{x}, \lambda(t)).
\end{aligned}$$

Using again that $(x(t), \lambda(t))$ remains bounded on $[t_0, +\infty[$, and owing to the fact that $T_t(x(t), \lambda(t)) \to (0, 0)$ strongly in $X \times Y$ as $t \to +\infty$ under each of the above conditions on $\varepsilon(t)$, passing to the limit entails

$$\lim_{t\to+\infty} \big(L(x(t), \bar{\lambda}) - L(\bar{x}, \lambda(t))\big) = 0,$$

concluding the result. □

Remark 3.5. Let us compare the estimates derived in the proof of Theorem 3.4 in the "limiting case" when $\varepsilon(t) = 1/t$ with $t_0 > 0$. On the one hand, for every $t \geq t_0$, we have

$$\begin{aligned}
\sqrt{\vartheta(t)} &\leq \mathrm{e}^{-\rho(t)}\,\sqrt{\vartheta(t_0)} + \frac{K}{\sqrt{2}}\,\mathrm{e}^{-\rho(t)} \int_{t_0}^{t} \mathrm{e}^{\rho(\tau)}|\dot{\varepsilon}(\tau)|\,\mathrm{d}\tau \\
&= \sqrt{\vartheta(t_0)}\frac{t_0}{t} + \frac{K}{\sqrt{2}t}\ln\Big(\frac{t}{t_0}\Big).
\end{aligned}$$

Consequently, $\vartheta(t)$ obeys the asymptotic estimate

$$\vartheta(t) = \mathcal{O}\Big(\frac{\ln(t)^2}{t^2}\Big) \text{ as } t \to +\infty.$$

On the other hand, for every $t \geq t_0$, we have

$$\begin{aligned}\vartheta(t) &\leq \mathrm{e}^{-\rho(t)}\,\vartheta(t_0) + \frac{K}{2}\,\mathrm{e}^{-\rho(t)} \int_{t_0}^{t} \mathrm{e}^{\rho(\tau)}\,\frac{|\dot{\varepsilon}(\tau)|^2}{\varepsilon(\tau)}\,\mathrm{d}\tau \\ &= \vartheta(t_0)\frac{t_0}{t} + \frac{K}{2t^2}\Big(\frac{t}{t_0} - 1\Big).\end{aligned}$$

In this case, we obtain the comparable decay rate estimate

$$\vartheta(t) = \mathcal{O}\Big(\frac{1}{t}\Big) \text{ as } t \to +\infty.$$

Remark 3.6. We note that the above result under the condition $\dot{\varepsilon} \in \mathcal{L}^1([t_0, +\infty[)$ has already been established by Cominetti et al. [17, Theorem 9]) using a similar line of arguments. In the recent work of Boţ and Nguyen [12, Theorem 2.5] it has been shown that $(\dot{x}(t), \dot{\lambda}(t))$ also tends to zero as $t \to +\infty$ whenever

$$\lim_{t\to+\infty} \frac{|\dot{\varepsilon}(t)|}{\varepsilon(t)} = 0.$$

We are now in the position to assert the strong convergence of the solutions of the (AHT) differential system.

Proposition 3.7. *Let $S \times M$ be non-empty, let $(x, \lambda) : [t_0, +\infty[\to X \times Y$ be a solution of* (AHT)*, and suppose that $\varepsilon \notin \mathcal{L}^1([t_0, +\infty[)$ with either $\dot{\varepsilon} \in \mathcal{L}^1([t_0, +\infty[)$ or $|\dot{\varepsilon}|^2/\varepsilon \in \mathcal{L}^1([t_0, +\infty[)$. Then it holds that*

$$\lim_{t\to+\infty} (x(t), \lambda(t)) = \mathrm{proj}_{S\times M}(0, 0).$$

Proof. In view of Corollary 3.2 (see also Remark 3.3), it suffices to show that

$$\liminf_{t\to+\infty} \|(x(t), \lambda(t))\| \geq \|\mathrm{proj}_{S\times M}(0, 0)\|.$$

Let $(x, \lambda) \in X \times Y$ and suppose that $(x(t_n), \lambda(t_n)) \rightharpoonup (\bar{x}, \bar{\lambda})$ weakly in $X \times Y$, as $n \to +\infty$, for a sequence $t_n \to +\infty$. By virtue of Theorem 3.4, we have

$$\begin{aligned}0 &= \lim_{n\to+\infty} \big(L(x(t_n), \lambda) - L(x, \lambda(t_n))\big) \\ &\geq \liminf_{n\to+\infty}\, L(x(t_n), \lambda) + \liminf_{n\to+\infty} (-L(x, \lambda(t_n))) \\ &\geq L(\bar{x}, \lambda) - L(x, \bar{\lambda})\end{aligned}$$

thanks to the weak lower semi-continuity of $L(\,\cdot\,, \lambda)$ and $-L(x, \cdot\,)$. The above inequalities being true for every $(x, \lambda) \in X \times Y$, we conclude that $(\bar{x}, \bar{\lambda}) \in S \times M$.

On the other hand, the weak lower semi-continuity of the norm $\|\cdot\|$ ensures

$$\liminf_{n\to+\infty} \|(x(t_n), \lambda(t_n))\| \geq \|(\bar{x}, \bar{\lambda})\|.$$

This inequality being true for every $(\bar{x}, \bar{\lambda}) \in S\times M$, taking the minimum over $S\times M$ yields the desired conclusion. □

Let us next provide a strong convergence result for the solutions of (AHT) under the assumption that $\varepsilon(t)$ is twice continuously differentiable with

$$\lim_{t\to+\infty} \varepsilon(t) = 0.$$

Proposition 3.8. *Let $S \times M$ be non-empty, let $(x,\lambda) : [t_0,+\infty[\to X \times Y$ be a solution of* (AHT)*, and let $\varepsilon : [t_0,+\infty[\to]0,+\infty[$ be such that*

$$\left.\begin{aligned} \varepsilon^2(t) + \dot\varepsilon(t) \geq 0 \\ 2\varepsilon(t)\dot\varepsilon(t) + \ddot\varepsilon(t) \leq 0 \end{aligned}\right\} \quad \forall t \geq t_+$$

for some $t_+ \geq t_0$. Then it holds that

$$\lim_{t\to+\infty} (x(t),\lambda(t)) = \mathrm{proj}_{S\times M}(0,0).$$

Proof. By virtue of Cominetti et al. [17, Proposition 6], it suffices to show that $\varepsilon \notin \mathcal{L}^1([t_+,+\infty[)$ and that all weak sequential cluster points of $(x(t),\lambda(t))_{t\geq t_0}$ belong to the set $S\times M$. Let $t_+ \geq t_0$ be such that $1 \geq -\dot\varepsilon(t)/\varepsilon^2(t)$ for all $t \geq t_+$. An immediate integration over $[t_+,t]$ yields

$$t - t_+ \geq \frac{1}{\varepsilon(t)} - \frac{1}{\varepsilon(t_+)}.$$

Successively integrating again and passing to the limit as $t \to +\infty$ entails

$$\int_{t_+}^{\infty} \varepsilon(\tau)\,\mathrm{d}\tau \geq \int_{t_+}^{\infty} \frac{1}{\tau - t_+ + \frac{1}{\varepsilon(t_+)}}\,\mathrm{d}\tau = +\infty$$

so that $\varepsilon \notin \mathcal{L}^1([t_+,+\infty[)$.

Let $(x,\lambda) \in X\times Y$ and suppose now that $(x(t_n),\lambda(t_n)) \rightharpoonup (\bar x,\bar\lambda)$ weakly in $X\times Y$, as $n\to+\infty$, for a sequence $t_n \to +\infty$. Since $(x(t),\lambda(t))$ is bounded on $[t_0,+\infty[$ and $(\dot x(t),\dot\lambda(t)) \to (0,0)$ strongly in $X\times Y$ as $t\to+\infty$ (as it will be justified later in Proposition 4.2), it follows from inequality (3.1) together with the regularized Lagrangian L_t that

$$\begin{aligned} 0 &\geq \limsup_{n\to+\infty} \big(L(x_{t_n},\lambda) - L(x,\lambda_{t_n})\big) \\ &\geq \liminf_{n\to+\infty} L(x_{t_n},\lambda) + \liminf_{n\to+\infty} (-L(x,\lambda_{t_n})) \\ &\geq L(\bar x,\lambda) - L(x,\bar\lambda), \end{aligned}$$

where we again utilized the weak lower semi-continuity of $L(\,\cdot\,,\lambda)$ and $-L(x,\,\cdot\,)$. The above derivations being true for every $(x,\lambda)\in X\times Y$, we conclude that $(\bar x,\bar\lambda)$ is a saddle point of L, that is, $(\bar x,\bar\lambda) \in S\times M$. ☐

Remark 3.9. Under the hypotheses of Proposition 3.7 or Proposition 3.8, for every $(\xi,\eta)\in X\times Y$, we immediately observe that the solutions $(x,\lambda) : [t_0,+\infty[\to X\times Y$ of the nonautonomous differential system

$$\begin{cases} \dot x + \nabla f(x) + A^*\lambda + \varepsilon(t)(x-\xi) = 0 \\ \dot\lambda + b - Ax + \varepsilon(t)(\lambda - \eta) = 0 \end{cases}$$

strongly converge toward $\mathrm{proj}_{S\times M}(\xi,\eta)$ as $t\to+\infty$.

4. Convergence rate estimates

In this section, we aim at deriving fast convergence rate estimates for the (AHT) solutions. To this end, we again restrict the class of Tikhonov regularization functions by replacing assumption (A4) such that

(A4)′ $\varepsilon : [t_0,+\infty[\to]0,+\infty[$ is twice continuously differentiable with

$$\lim_{t\to+\infty} \varepsilon(t) = 0.$$

4.1. **Asymptotics relative to the set of zeros.** Let us begin our discussion by deriving fast decay rate estimates for the solutions of the (AHT) differential system with respect to its set of zeros. The following result is based on the assumption that the Tikhonov regularization function $\varepsilon : [t_0, +\infty[\to]0, +\infty[$ verifies the conditions

$$\left.\begin{aligned} \varepsilon^2(t) + \dot{\varepsilon}(t) \geq 0 \\ 2\varepsilon(t)\dot{\varepsilon}(t) + \ddot{\varepsilon}(t) \leq 0 \end{aligned}\right\} \quad \forall t \geq t_+$$

for some $t_+ \geq t_0$.

Theorem 4.1. *Let $S \times M$ be non-empty, let $(x, \lambda) : [t_0, +\infty[\to X \times Y$ be a solution of* (AHT)*, and let $\varepsilon : [t_0, +\infty[\to]0, +\infty[$ be such that*

$$\left.\begin{aligned} \varepsilon^2(t) + \dot{\varepsilon}(t) \geq 0 \\ 2\varepsilon(t)\dot{\varepsilon}(t) + \ddot{\varepsilon}(t) \leq 0 \end{aligned}\right\} \quad \forall t \geq t_+$$

for some $t_+ \geq t_0$. Then, for every $(\bar{x}, \bar{\lambda}) \in S \times M$, the following assertions hold:

$$\begin{aligned} \|(\dot{x}(t), \dot{\lambda}(t)) + \varepsilon(t)((x(t), \lambda(t)) - (\bar{x}, \bar{\lambda}))\|^2 &= \mathcal{O}\big(\mathrm{e}^{-2\rho(t)} + \varepsilon^2(t)\big) \ \text{ as } t \to +\infty; \\ \varepsilon(t)\big(L(x(t), \bar{\lambda}) - L(\bar{x}, \lambda(t))\big) &= \mathcal{O}\big(\mathrm{e}^{-2\rho(t)} + \varepsilon^2(t)\big) \ \text{ as } t \to +\infty; \\ \|T(x(t), \lambda(t)) - T(\bar{x}, \bar{\lambda})\|^2 &= \mathcal{O}\big(\mathrm{e}^{-2\rho(t)} + \varepsilon^2(t)\big) \ \text{ as } t \to +\infty. \end{aligned}$$

Proof. Let $(\bar{x}, \bar{\lambda}) \in S \times M$ and define $\psi : [t_0, +\infty[\to \mathbb{R}$ by $\psi(t) = \|(\dot{x}(t), \dot{\lambda}(t)) + \varepsilon(t)((x(t), \lambda(t)) - (\bar{x}, \bar{\lambda}))\|^2/2$. Moreover, let $\sigma : [t_0, +\infty[\to \mathbb{R}$ be defined by $\sigma(t) = \varepsilon^2(t) + \dot{\varepsilon}(t)$ such that $\dot{\sigma}(t) = 2\varepsilon(t)\dot{\varepsilon}(t) + \ddot{\varepsilon}(t)$. Differentiating $\psi(t)$ and taking (AHT) into account yields, for almost every $t \geq t_0$,

$$\dot{\psi}(t) + \Big\langle \frac{\mathrm{d}}{\mathrm{d}t} T(x(t), \lambda(t)) + \dot{\varepsilon}(t)(\bar{x}, \bar{\lambda}), (\dot{x}(t), \dot{\lambda}(t)) + \varepsilon(t)((x(t), \lambda(t)) - (\bar{x}, \bar{\lambda})) \Big\rangle = 0.$$

In view of an immediate expansion, we obtain

$$\begin{aligned} &\Big\langle \frac{\mathrm{d}}{\mathrm{d}t} T(x(t), \lambda(t)) + \dot{\varepsilon}(t)(\bar{x}, \bar{\lambda}), (\dot{x}(t), \dot{\lambda}(t)) + \varepsilon(t)((x(t), \lambda(t)) - (\bar{x}, \bar{\lambda})) \Big\rangle \\ &\quad + \big\langle T(x(t), \lambda(t)) + \varepsilon(t)(\bar{x}, \bar{\lambda}), (\dot{x}(t), \dot{\lambda}(t)) + \varepsilon(t)((x(t), \lambda(t)) - (\bar{x}, \bar{\lambda})) \big\rangle \\ &\quad + \dot{\psi}(t) + 2\varepsilon(t)\psi(t) = 0. \end{aligned}$$

Utilizing the basic identity

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\big(\varepsilon(t)\langle T(x(t), \lambda(t)), (x(t), \lambda(t)) - (\bar{x}, \bar{\lambda})\rangle\big) &= \varepsilon(t)\langle T(x(t), \lambda(t)), (\dot{x}(t), \dot{\lambda}(t))\rangle \\ + \varepsilon(t)\Big\langle \frac{\mathrm{d}}{\mathrm{d}t} T(x(t), \lambda(t)), (x(t), \lambda(t)) - (\bar{x}, \bar{\lambda}) \Big\rangle &+ \dot{\varepsilon}(t)\langle T(x(t), \lambda(t)), (x(t), \lambda(t)) - (\bar{x}, \bar{\lambda})\rangle \end{aligned}$$

together with the fact that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\big(\dot{\varepsilon}(t)\langle (x(t), \lambda(t)) - (\bar{x}, \bar{\lambda}), (\bar{x}, \bar{\lambda})\rangle\big) &= \ddot{\varepsilon}(t)\langle (x(t), \lambda(t)) - (\bar{x}, \bar{\lambda}), (\bar{x}, \bar{\lambda})\rangle \\ &+ \dot{\varepsilon}(t)\frac{\mathrm{d}}{\mathrm{d}t}\langle (x(t), \lambda(t)) - (\bar{x}, \bar{\lambda}), (\bar{x}, \bar{\lambda})\rangle, \end{aligned}$$

in view of the simple expansion

$$\begin{aligned} \langle (x(t), \lambda(t)) - (\bar{x}, \bar{\lambda}), (\bar{x}, \bar{\lambda})\rangle &= \frac{1}{2}\|(x(t), \lambda(t))\|^2 - \frac{1}{2}\|(\bar{x}, \bar{\lambda})\|^2 \\ &- \frac{1}{2}\|(x(t), \lambda(t)) - (\bar{x}, \bar{\lambda})\|^2, \end{aligned}$$

the above equality reads as

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\psi(t)+\varepsilon(t)\langle T(x(t),\lambda(t)),(x(t),\lambda(t))-(\bar{x},\bar{\lambda})\rangle+\frac{\sigma(t)}{2}\|(x(t),\lambda(t))\|^2-\frac{\varepsilon^2(t)}{2}\|(\bar{x},\bar{\lambda})\|^2\Big)$$
$$+2\varepsilon(t)\Big(\psi(t)+\varepsilon(t)\langle T(x(t),\lambda(t)),(x(t),\lambda(t))-(\bar{x},\bar{\lambda})\rangle+\frac{\sigma(t)}{2}\|(x(t),\lambda(t))\|^2-\frac{\varepsilon^2(t)}{2}\|(\bar{x},\bar{\lambda})\|^2\Big)$$
$$+\Big\langle\frac{\mathrm{d}}{\mathrm{d}t}T(x(t),\lambda(t)),(\dot{x}(t),\dot{\lambda}(t))\Big\rangle-\frac{\dot{\sigma}(t)}{2}\|(x(t),\lambda(t))\|^2=0.$$

Multiplying by $\mathrm{e}^{2\rho(t)}$ and taking into account that T is monotone entails

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{e}^{2\rho(t)}\left(\psi(t)+\varepsilon(t)\langle T(x(t),\lambda(t)),(x(t),\lambda(t))-(\bar{x},\bar{\lambda})\rangle\right)\Big)$$
$$+\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{e}^{2\rho(t)}\Big(\frac{\sigma(t)}{2}\|(x(t),\lambda(t))\|^2-\frac{\varepsilon^2(t)}{2}\|(\bar{x},\bar{\lambda})\|^2\Big)\Big)$$
$$-\mathrm{e}^{2\rho(t)}\frac{\dot{\sigma}(t)}{2}\|(x(t),\lambda(t))\|^2\le 0.$$

Integration over $[t_+,t]$ and subsequently dividing by $\mathrm{e}^{2\rho(t)}$ shows that there exists $K\ge 0$ such that

$$\begin{aligned}&\psi(t)+\varepsilon(t)\langle T(x(t),\lambda(t)),(x(t),\lambda(t))-(\bar{x},\bar{\lambda})\rangle+\frac{\sigma(t)}{2}\|(x(t),\lambda(t))\|^2\\&-\mathrm{e}^{-2\rho(t)}\int_{t_+}^{t}\mathrm{e}^{2\rho(\tau)}\frac{\dot{\sigma}(\tau)}{2}\|(x(\tau),\lambda(\tau))\|^2\,\mathrm{d}\tau\le K\,\mathrm{e}^{-2\rho(t)}+\frac{\varepsilon^2(t)}{2}\|(\bar{x},\bar{\lambda})\|^2.\end{aligned}\tag{4.1}$$

Noticing that we have both $\sigma(t)\ge 0$ and $\dot{\sigma}(t)\le 0$ for all $t\ge t_+$, we deduce

$$\psi(t)+\varepsilon(t)\langle T(x(t),\lambda(t)),(x(t),\lambda(t))-(\bar{x},\bar{\lambda})\rangle$$
$$\le K\,\mathrm{e}^{-2\rho(t)}+\frac{\varepsilon^2(t)}{2}\|(\bar{x},\bar{\lambda})\|^2.$$

Now, taking into account that $(\bar{x},\bar{\lambda})$ is a saddle point of L and using the fact that

$$\langle T(x(t),\lambda(t)),(x(t),\lambda(t))-(\bar{x},\bar{\lambda})\rangle\ge L(x(t),\bar{\lambda})-L(\bar{x},\lambda(t)),$$

we obtain both

$$\psi(t)\le K\,\mathrm{e}^{-2\rho(t)}+\frac{\varepsilon^2(t)}{2}\|(\bar{x},\bar{\lambda})\|^2,\ \text{and}$$
$$\varepsilon(t)\big(L(x(t),\bar{\lambda})-L(\bar{x},\lambda(t))\big)\le K\,\mathrm{e}^{-2\rho(t)}+\frac{\varepsilon^2(t)}{2}\|(\bar{x},\bar{\lambda})\|^2.$$

Successively dividing the above inequalities by $\mathrm{e}^{-2\rho(t)}+\varepsilon^2(t)$ and passing to the upper limit as $t\to+\infty$ gives the desired estimates.

On the other hand, in view of the basic inequality

$$\frac{1}{2}\|(\dot{x}(t),\dot{\lambda}(t))+\varepsilon(t)(x(t),\lambda(t))\|^2\le 2\psi(t)+\varepsilon^2(t)\|(\bar{x},\bar{\lambda})\|^2,$$

it readily follows from (AHT) together with $T(\bar{x},\bar{\lambda})=(0,0)$ that

$$\frac{1}{2}\|T(x(t),\lambda(t))-T(\bar{x},\bar{\lambda})\|^2\le 2\psi(t)+\varepsilon^2(t)\|(\bar{x},\bar{\lambda})\|^2.$$

Dividing again by $\mathrm{e}^{-2\rho(t)}+\varepsilon^2(t)$ and passing to the upper limit as $t\to+\infty$ concludes the result. □

As an immediate consequence of the above result, we recover the decay rate estimate

$$\|(\dot{x}(t), \dot{\lambda}(t))\|^2 = \mathcal{O}\big(\mathrm{e}^{-2\rho(t)} + |\dot{\varepsilon}(t)|\big) \text{ as } t \to +\infty$$

recently obtained by Boţ and Nguyen [12, Theorem 2.7] in the context of general monotone operator flows with Tikhonov regularization.

Proposition 4.2. *Let $S \times M$ be non-empty, let $(x, \lambda) : [t_0, +\infty[\to X \times Y$ be a solution of* (AHT)*, and let $\varepsilon : [t_0, +\infty[\to]0, +\infty[$ be such that*

$$\left.\begin{aligned} \varepsilon^2(t) + \dot{\varepsilon}(t) \geq 0 \\ 2\varepsilon(t)\dot{\varepsilon}(t) + \ddot{\varepsilon}(t) \leq 0 \end{aligned}\right\} \quad \forall t \geq t_+$$

for some $t_+ \geq t_0$. Then it holds that

$$\|(\dot{x}(t), \dot{\lambda}(t))\|^2 = \mathcal{O}\big(\mathrm{e}^{-2\rho(t)} + \varepsilon^2(t)\big) \text{ as } t \to +\infty.$$

Proof. Let $(\bar{x}, \bar{\lambda}) \in S \times M$ and consider again $\vartheta : [t_0, +\infty[\to \mathbb{R}$ defined by $\vartheta(t) = \|(\dot{x}(t), \dot{\lambda}(t))\|^2/2$. Recall from inequality (4.1) that for every $t \geq t_+$, we have

$$\begin{aligned} \psi(t) + \varepsilon(t)\langle T(x(t), \lambda(t)), (x(t), \lambda(t)) - (\bar{x}, \bar{\lambda})\rangle + \frac{\sigma(t)}{2}\|(x(t), \lambda(t))\|^2 \\ - \mathrm{e}^{-2\rho(t)} \int_{t_+}^{t} \mathrm{e}^{2\rho(\tau)} \frac{\dot{\sigma}(\tau)}{2}\|(x(\tau), \lambda(\tau))\|^2 \,\mathrm{d}\tau \leq K \,\mathrm{e}^{-2\rho(t)} + \frac{\varepsilon^2(t)}{2}\|(\bar{x}, \bar{\lambda})\|^2. \end{aligned}$$

Utilizing the fact that $\dot{\sigma}(t) \leq 0$ for all $t \geq t_+$, we obtain

$$\begin{aligned} \psi(t) + \varepsilon(t)\langle T(x(t), \lambda(t)), (x(t), \lambda(t)) - (\bar{x}, \bar{\lambda})\rangle + \frac{\sigma(t)}{2}\|(x(t), \lambda(t))\|^2 \\ \leq K \,\mathrm{e}^{-2\rho(t)} + \frac{\varepsilon^2(t)}{2}\|(\bar{x}, \bar{\lambda})\|^2. \end{aligned}$$

Equivalently, in view of (AHT), the above inequality reads

$$\vartheta(t) + \frac{\sigma(t)}{2}\|(x(t), \lambda(t))\|^2 \leq K \,\mathrm{e}^{-2\rho(t)} + \frac{\varepsilon^2(t)}{2}\|(x(t), \lambda(t))\|^2.$$

Taking into account that $\sigma(t) \geq 0$ for all $t \geq t_+$, we have

$$\vartheta(t) \leq K \,\mathrm{e}^{-2\rho(t)} + \frac{\varepsilon^2(t)}{2}\|(x(t), \lambda(t))\|^2.$$

Since $(x(t), \lambda(t))$ remains bounded on $[t_0, +\infty[$, dividing by $\mathrm{e}^{-2\rho(t)} + \varepsilon^2(t)$ and passing to the upper limit as $t \to +\infty$ yields the desired conclusion. □

The following result provides a more refined estimate for the velocity of an (AHT) solution given the additional assumption that the Tikhonov regularization function $\varepsilon : [t_0, +\infty[\to]0, +\infty[$ is decreasing.

Proposition 4.3. *Let $S \times M$ be non-empty, let $(x, \lambda) : [t_0, +\infty[\to X \times Y$ be a solution of* (AHT)*, and let $\varepsilon : [t_0, +\infty[\to]0, +\infty[$ be decreasing such that*

$$\left.\begin{aligned} \varepsilon^2(t) + \dot{\varepsilon}(t) \geq 0 \\ 2\varepsilon(t)\dot{\varepsilon}(t) + \ddot{\varepsilon}(t) \leq 0 \end{aligned}\right\} \quad \forall t \geq t_+$$

for some $t_+ \geq t_0$. Then the following assertion holds:

$$\lim_{t \to +\infty} \mathrm{e}^{-\rho(t)} \int_{t_+}^{t} \mathrm{e}^{\rho(\tau)} \frac{1}{\varepsilon(\tau)}\|(\dot{x}(\tau), \dot{\lambda}(\tau))\|^2 \,\mathrm{d}\tau < +\infty.$$

Proof. Let $(\bar{x},\bar{\lambda}) \in S \times M$ and consider again $\phi : [t_0, +\infty[\to \mathbb{R}$ defined by $\phi(t) = \|(x(t),\lambda(t)) - (\bar{x},\bar{\lambda})\|^2/2$. Observing that we have both $\sigma(t) \geq 0$ and $\dot{\sigma}(t) \leq 0$ for all $t \geq t_+$, it follows from inequality (4.1) and the monotonicity of T that

$$\psi(t) \leq K\,\mathrm{e}^{-2\rho(t)} + \frac{\varepsilon^2(t)}{2}\|(\bar{x},\bar{\lambda})\|^2.$$

Equivalently, we have

$$\varepsilon(t)\dot{\phi}(t) + \varepsilon^2(t)\phi(t) + \vartheta(t) \leq K\,\mathrm{e}^{-2\rho(t)} + \frac{\varepsilon^2(t)}{2}\|(\bar{x},\bar{\lambda})\|^2.$$

Successively dividing by $\varepsilon(t)$ and multiplying by $\mathrm{e}^{\rho(t)}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{e}^{\rho(t)}\,\phi(t)\big) + \mathrm{e}^{\rho(t)}\,\frac{1}{\varepsilon(t)}\vartheta(t) \leq \frac{K}{\varepsilon(t)}\,\mathrm{e}^{-\rho(t)} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{\rho(t)}\|(\bar{x},\bar{\lambda})\|^2.$$

Integration over $[t_+, t]$ and subsequently dividing by $\mathrm{e}^{\rho(t)}$ entails

$$\begin{aligned}\phi(t) + \mathrm{e}^{-\rho(t)}\int_{t_+}^{t}\mathrm{e}^{\rho(\tau)}\,\frac{1}{\varepsilon(\tau)}\vartheta(\tau)\,\mathrm{d}\tau &\leq K\,\mathrm{e}^{-\rho(t)}\int_{t_+}^{t}\frac{1}{\varepsilon(\tau)}\,\mathrm{e}^{-\rho(\tau)}\,\mathrm{d}\tau \\ &\quad + \mathrm{e}^{-(\rho(t)-\rho(t_+))}\,\phi(t_+) + \frac{1}{2}\|(\bar{x},\bar{\lambda})\|^2.\end{aligned} \tag{4.2}$$

On the other hand, since $\dot{\sigma}(t) \leq 0$ for all $t \geq t_+$, it follows from an immediate integration that

$$-\dot{\varepsilon}(t)\,\mathrm{e}^{2\rho(t)} \geq -\dot{\varepsilon}(t_+)\,\mathrm{e}^{2\rho(t_+)}.$$

Owing to the fact that $\varepsilon(t)$ is decreasing, we obtain

$$-\frac{1}{\dot{\varepsilon}(t_+)\,\mathrm{e}^{2\rho(t_+)}}\frac{-\dot{\varepsilon}(t)}{\varepsilon(t)} \geq \frac{1}{\varepsilon(t)}\,\mathrm{e}^{-2\rho(t)}.$$

Multiplying by $\mathrm{e}^{\rho(t)}$ and taking into account that $\sigma(t) \geq 0$ for all $t \geq t_+$ yields

$$-\frac{1}{\dot{\varepsilon}(t_+)\,\mathrm{e}^{2\rho(t_+)}}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{\rho(t)} \geq \frac{1}{\varepsilon(t)}\,\mathrm{e}^{-\rho(t)}.$$

Integration over $[t_+, t]$ and subsequently dividing by $\mathrm{e}^{\rho(t)}$ gives

$$-\frac{1}{\dot{\varepsilon}(t_+)\,\mathrm{e}^{2\rho(t_+)}} \geq \mathrm{e}^{-\rho(t)}\int_{t_+}^{t}\frac{1}{\varepsilon(\tau)}\,\mathrm{e}^{-\rho(\tau)}\,\mathrm{d}\tau,$$

implying that the integral on the right-hand side of inequality (4.2) remains bounded on $[t_+, +\infty[$. Taking into account that $\phi(t) \geq 0$, passing to the limit in inequality (4.2) as $t \to +\infty$ then gives the desired result. □

Let us now investigate the convergence rate estimates for the (AHT) solutions under the assumption that there exists $\alpha > 0$ such that for every $(x,\lambda),(\xi,\eta) \in X \times Y$,

$$\|T(x,\lambda) - T(\xi,\eta)\|^2 \geq \alpha\|(x,\lambda) - (\xi,\eta)\|^2. \tag{L}$$

Recall that condition (L) clearly implies that the set $S \times M$ is a singleton. The following result is an immediate consequence of Theorem 4.1.

Corollary 4.4. *Under the hypotheses of Theorem 4.1, suppose that $T : X \times Y \to X \times Y$ satisfies condition* (L)*. Then, for $(\bar{x},\bar{\lambda}) \in S \times M$, it holds that*

$$\begin{aligned}\|(x(t),\lambda(t)) - (\bar{x},\bar{\lambda})\|^2 &= \mathcal{O}\big(\mathrm{e}^{-2\rho(t)} + \varepsilon^2(t)\big) \text{ as } t \to +\infty; \\ \|(\dot{x}(t),\dot{\lambda}(t))\|^2 &= \mathcal{O}\big(\big(1+\varepsilon^2(t)\big)\big(e^{-2\rho(t)} + \varepsilon^2(t)\big)\big) \text{ as } t \to +\infty.\end{aligned}$$

Proof. Let $(\bar{x}, \bar{\lambda}) \in S \times M$ and consider again $\phi : [t_0, +\infty[\to \mathbb{R}$ defined by $\phi(t) = \|(x(t), \lambda(t)) - (\bar{x}, \bar{\lambda})\|^2/2$. Since T satisfies condition (L), there exists $\alpha > 0$ such that for every $t \geq t_+$, it holds that

$$2\alpha\phi(t) \leq \|T(x(t), \lambda(t)) - T(\bar{x}, \bar{\lambda})\|^2.$$

The desired estimate now follows at once from Theorem 4.1.

On the other hand, from inequality (4.1), we observe that for every $t \geq t_+$, it holds that

$$\psi(t) \leq K\,\mathrm{e}^{-2\rho(t)} + \frac{\varepsilon^2(t)}{2}\|(\bar{x}, \bar{\lambda})\|^2.$$

The assertion now readily follows by combining the basic inequality

$$\vartheta(t) \leq 2\psi(t) + 2\varepsilon^2(t)\phi(t)$$

with the above estimate. □

4.2. **Asymptotics relative to the viscosity curve.** Let us now adapt the previous results to obtain fast decay rate estimates for the (AHT) solutions with respect to the viscosity curve (x_t, λ_t). Recall that the viscosity curve (x_t, λ_t) is characterized, for each $t \geq t_0$, as the unique zero of the $\varepsilon(t)$-strongly monotone operator

$$T_t = T + \varepsilon(t)\,\mathrm{Id}\,.$$

The following result is analog to Theorem 4.1.

Theorem 4.5. *Let $S \times M$ be non-empty and let $(x, \lambda) : [t_0, +\infty[\to X \times Y$ be a solution of* (AHT)*. Let $(x_t, \lambda_t) = \operatorname{zer} T_t$ for each $t \geq t_0$ and suppose that $\varepsilon : [t_0, +\infty[\to]0, +\infty[$ verifies*

$$\left.\begin{aligned} \varepsilon^2(t) + \dot{\varepsilon}(t) &\geq 0 \\ 2\varepsilon(t)\dot{\varepsilon}(t) + \ddot{\varepsilon}(t) &\leq 0 \end{aligned}\right\} \quad \forall t \geq t_+$$

for some $t_+ \geq t_0$. Then the following assertions hold:

$$\|(\dot{x}(t), \dot{\lambda}(t)) + \varepsilon(t)((x(t), \lambda(t)) - (x_t, \lambda_t))\|^2 = \mathcal{O}\big(\mathrm{e}^{-2\rho(t)} + \varepsilon^2(t)\big) \text{ as } t \to +\infty;$$
$$\|T(x(t), \lambda(t)) - T(x_t, \lambda_t)\|^2 = \mathcal{O}\big(\mathrm{e}^{-2\rho(t)} + \varepsilon^2(t)\big) \text{ as } t \to +\infty.$$

Proof. Let $S \times M$ be non-empty, let $(x_t, \lambda_t) = \operatorname{zer} T_t$, and let $\theta : [t_0, +\infty[\to \mathbb{R}$ be defined by $\theta(t) = \|(\dot{x}(t), \dot{\lambda}(t) + \varepsilon(t)((x(t), \lambda(t)) - (x_t, \lambda_t))\|^2/2$. Consider again $\sigma : [t_0, +\infty[\to \mathbb{R}$ defined by $\sigma(t) = \varepsilon^2(t) + \dot{\varepsilon}(t)$ such that $\dot{\sigma}(t) = 2\varepsilon(t)\dot{\varepsilon}(t) + \ddot{\varepsilon}(t)$. Using similar derivations as in the proof of Theorem 4.1 shows that there exists $K \geq 0$ such that for every $t \geq t_+$, it holds that

$$\begin{aligned} \theta(t) + \varepsilon(t)\langle T(x(t), \lambda(t)), (x(t), \lambda(t)) - (x_t, \lambda_t)\rangle + \frac{\sigma(t)}{2}\|(x(t), \lambda(t))\|^2 \\ - \mathrm{e}^{-2\rho(t)} \int_{t_+}^{t} \mathrm{e}^{2\rho(\tau)}\,\frac{\dot{\sigma}(\tau)}{2}\|(x(\tau), \lambda(\tau))\|^2\,\mathrm{d}\tau \leq K\,\mathrm{e}^{-2\rho(t)} + \frac{\varepsilon^2(t)}{2}\|(x_t, \lambda_t)\|^2. \end{aligned}$$

Observing that we have both $\sigma(t) \geq 0$ and $\dot{\sigma}(t) \leq 0$ for all $t \geq t_+$, we infer

$$\begin{aligned} \theta(t) + \varepsilon(t)\langle T(x(t), \lambda(t)), (x(t), \lambda(t)) - (x_t, \lambda_t)\rangle \\ \leq K\,\mathrm{e}^{-2\rho(t)} + \frac{\varepsilon^2(t)}{2}\|(x_t, \lambda_t)\|^2. \end{aligned}$$

In view of the system of primal-dual optimality conditions

$$T(x_t, \lambda_t) + \varepsilon(t)(x_t, \lambda_t) = (0, 0),$$

we readily deduce

$$\begin{aligned}\theta(t) + \varepsilon(t)\langle T(x(t),\lambda(t)) - T(x_t,\lambda_t), (x(t),\lambda(t)) - (x_t,\lambda_t)\rangle \\ + \varepsilon^2(t)\zeta(t) \le K\,\mathrm{e}^{-2\rho(t)} + \frac{\varepsilon^2(t)}{2}\|(x(t),\lambda(t))\|^2\end{aligned}$$

with $\zeta : [t_0,+\infty[\; \to \mathbb{R}$ being defined as $\zeta(t) = \|(x(t),\lambda(t)) - (x_t,\lambda_t)\|^2/2$. Upon using the fact that T is monotone and taking into account that $(x(t),\lambda(t))$ remains bounded on $[t_0,+\infty[$, successively dividing by $\mathrm{e}^{-2\rho(t)} + \varepsilon^2(t)$ and passing to the upper limit gives the desired estimates. □

Similarly to Proposition 4.3, some more refined estimates can be derived under the additional assumption that the Tikhonov regularization function $\varepsilon : [t_0,+\infty[\; \to\;]0,+\infty[$ is decreasing.

Proposition 4.6. *Let $S \times M$ be non-empty and let $(x,\lambda) : [t_0,+\infty[\; \to X \times Y$ be a solution of* (AHT). *Let $(x_t,\lambda_t) = \operatorname{zer} T_t$ for each $t \ge t_0$ and suppose that $\varepsilon : [t_0,+\infty[\; \to\;]0,+\infty[$ is decreasing such that*

$$\left.\begin{aligned}\varepsilon^2(t) + \dot\varepsilon(t) \ge 0\\ 2\varepsilon(t)\dot\varepsilon(t) + \ddot\varepsilon(t) \le 0\end{aligned}\right\} \quad \forall t \ge t_+$$

for some $t_+ \ge t_0$. Then the following estimates are verified:

$$\lim_{t\to+\infty} \mathrm{e}^{-\rho(t)} \int_{t_+}^{t} \mathrm{e}^{\rho(\tau)}\,\frac{1}{\varepsilon(\tau)}\|(\dot x(\tau),\dot\lambda(\tau)) + \varepsilon(t)((x(\tau),\lambda(\tau)) - (x_\tau,\lambda_\tau))\|^2\,\mathrm{d}\tau < +\infty;$$

$$\lim_{t\to+\infty} \mathrm{e}^{-\rho(t)} \int_{t_+}^{t} \mathrm{e}^{\rho(\tau)}\,\frac{1}{\varepsilon(\tau)}\|T(x(\tau),\lambda(\tau)) - T(x_\tau,\lambda_\tau)\|^2\,\mathrm{d}\tau < +\infty.$$

Proof. Let $S \times M$ be non-empty, let $(x_t,\lambda_t) = \operatorname{zer} T_t$, and let $\theta : [t_0,+\infty[\; \to \mathbb{R}$ be defined by $\theta(t) = \|(\dot x(t),\dot\lambda(t) + \varepsilon(t)((x(t),\lambda(t)) - (x_t,\lambda_t))\|^2/2$. In view of (AHT) and the system of primal-dual optimality conditions, for every $t \ge t_0$, it holds that

$$\begin{aligned}\|(\dot x(t),\dot\lambda(t))\|^2 &= \|T(x(t),\lambda(t)) - T(x_t,\lambda_t) + \varepsilon(t)((x(t),\lambda(t)) - (x_t,\lambda_t))\|^2\\ &\ge \|T(x(t),\lambda(t)) - T(x_t,\lambda_t)\|^2 + \varepsilon^2(t)\|(x(t),\lambda(t))\|^2\\ &\ge \|T(x(t),\lambda(t)) - T(x_t,\lambda_t)\|^2,\end{aligned} \tag{4.3}$$

where the first inequality follows from the monotonicity of T. Upon using (AHT) again, we infer

$$\theta(t) \le \frac{1}{2}\|(\dot x(t),\dot\lambda(t))\|^2.$$

The assertions are now readily deduced as in Proposition 4.3. □

Finally, let us provide some refined estimates for the (AHT) solutions assuming again that $T : X \times Y \to X \times Y$ verifies condition (L).

Corollary 4.7. *Under the hypotheses of Theorem 4.5, suppose that $T : X \times Y \to X \times Y$ satisfies condition* (L). *Then the following assertions hold:*

$$\|T(x(t),\lambda(t)) - T(x_t,\lambda_t)\|^2 = \mathcal{O}\big(\big(1+\varepsilon^2(t)\big)\big(e^{-2\rho(t)} + \varepsilon^2(t)\big)\big) \text{ as } t \to +\infty;$$

$$\|(x(t),\lambda(t)) - (x_t,\lambda_t)\|^2 = \mathcal{O}\big(\big(1+\varepsilon^2(t)\big)\big(e^{-2\rho(t)} + \varepsilon^2(t)\big)\big) \text{ as } t \to +\infty.$$

Proof. Let $S \times M$ be non-empty, let $(x_t, \lambda_t) = \operatorname{zer} T_t$, and let $\zeta : [t_0, +\infty[\to \mathbb{R}$ be again defined in terms of $\zeta(t) = \|(x(t), \lambda(t)) - (x_t, \lambda_t)\|^2/2$. Recall from inequality (4.3) that for every $t \geq t_0$, we have

$$\|T(x(t), \lambda(t)) - T(x_t, \lambda_t)\|^2 \leq \|(\dot{x}(t), \dot{\lambda}(t))\|^2.$$

Since T satisfies condition (L), there exists $\alpha > 0$ such that

$$\alpha\zeta(t) \leq \frac{1}{2}\|(\dot{x}(t), \dot{\lambda}(t))\|^2.$$

The assertions now follow at once as in Corollary 4.4. $\square$

4.3. **The particular case $\varepsilon(t) = 1/t^p$.** Let us now particularize the previous results to the case when the Tikhonov regularization function $\varepsilon : [t_0, +\infty[\to]0, +\infty[$ takes the form

$$\varepsilon(t) = \frac{1}{t^p} \quad \text{with } p \in]0, 1] \text{ and } t_0 > 0.$$

Since $\varepsilon(t)$ vanishes as $t \to +\infty$ with $\varepsilon \notin \mathcal{L}^1([0, +\infty[)$ and $\dot{\varepsilon} \in \mathcal{L}^1([0, +\infty[)$ for every $p \in]0, 1]$, we immediately deduce from Proposition 3.7 that the solutions $(x(t), \lambda(t))$ of (AHT) strongly converge toward $\operatorname{proj}_{S \times M}(0, 0)$ as $t \to +\infty$. Moreover, for every $t \geq t_0$, we have $\dot{\varepsilon}(t) = -p/t^{p+1}$ and $\ddot{\varepsilon}(t) = p(p+1)/t^{p+2}$ so that

$$\varepsilon^2(t) + \dot{\varepsilon}(t) = \frac{1}{t^{2p}} - \frac{p}{t^{p+1}};$$
$$2\varepsilon(t)\dot{\varepsilon}(t) + \ddot{\varepsilon}(t) = -\frac{2p}{t^{2p+1}} + \frac{p(p+1)}{t^{p+2}}.$$

In the case $p = 1$, we have both $\varepsilon^2(t) + \dot{\varepsilon}(t) = 0$ and $2\varepsilon(t)\dot{\varepsilon}(t) + \ddot{\varepsilon}(t) = 0$ for every $t \geq t_0$. On the other hand, whenever $p \in]0, 1[$, we readily obtain

$$\frac{1}{t^{2p}} - \frac{p}{t^{p+1}} \geq 0 \quad \iff \quad \sqrt[1-p]{p} \leq t;$$
$$-\frac{2p}{t^{2p+1}} + \frac{p(p+1)}{t^{p+2}} \leq 0 \quad \iff \quad \sqrt[1-p]{\frac{p+1}{2}} \leq t.$$

Consequently, for every $p \in]0, 1]$, there exists $t_+ = \max\{t_0, \sqrt[1-p]{(p+1)/2}\}$ such that

$$\left.\begin{aligned} \varepsilon^2(t) + \dot{\varepsilon}(t) \geq 0 \\ 2\varepsilon(t)\dot{\varepsilon}(t) + \ddot{\varepsilon}(t) \leq 0 \end{aligned}\right\} \quad \forall t \geq t_+,$$

implying that the hypotheses of Theorem 4.1 are verified. This immediately leads to the following assertion.

Proposition 4.8. *Let $S \times M$ be non-empty, let $(x, \lambda) : [t_0, +\infty[\to X \times Y$ be a solution of* (AHT)*, and let $\varepsilon : [t_0, +\infty[\to]0, +\infty[$ be defined by $\varepsilon(t) = 1/t^p$ with $p \in]0, 1]$ and $t_0 > 0$. Then, for every $(\bar{x}, \bar{\lambda}) \in S \times M$, it holds that*

$$\|(\dot{x}(t), \dot{\lambda}(t)) + \frac{1}{t^p}((x(t), \lambda(t)) - (\bar{x}, \bar{\lambda}))\|^2 = \mathcal{O}\Big(\mathrm{e}^{-2\rho(t)} + \frac{1}{t^{2p}}\Big) \text{ as } t \to +\infty;$$
$$\frac{1}{t^p}\big(L(x(t), \bar{\lambda}) - L(\bar{x}, \lambda(t))\big) = \mathcal{O}\Big(\mathrm{e}^{-2\rho(t)} + \frac{1}{t^{2p}}\Big) \text{ as } t \to +\infty;$$
$$\|T(x(t), \lambda(t)) - T(\bar{x}, \bar{\lambda})\|^2 = \mathcal{O}\Big(\mathrm{e}^{-2\rho(t)} + \frac{1}{t^{2p}}\Big) \text{ as } t \to +\infty;$$
$$\|(\dot{x}(t), \dot{\lambda}(t))\|^2 = \mathcal{O}\Big(\mathrm{e}^{-2\rho(t)} + \frac{1}{t^{2p}}\Big) \text{ as } t \to +\infty.$$

Moreover, $(x(t), \lambda(t))$ converges strongly to $\operatorname{proj}_{S \times M}(0, 0)$ *as $t \to +\infty$.*

Remark 4.9. In view of the above result, we readily observe that the fastest rate of convergence is achieved for the value $p = 1$. In this case, the above asymptotic estimates reduce to

$$\begin{aligned}
\|(\dot{x}(t), \dot{\lambda}(t)) + \frac{1}{t}((x(t), \lambda(t)) - (\bar{x}, \bar{\lambda}))\|^2 &= \mathcal{O}\Big(\frac{1}{t^2}\Big) \text{ as } t \to +\infty;\\
\frac{1}{t}\big(L(x(t), \bar{\lambda}) - L(\bar{x}, \lambda(t))\big) &= \mathcal{O}\Big(\frac{1}{t^2}\Big) \text{ as } t \to +\infty;\\
\|T(x(t), \lambda(t)) - T(\bar{x}, \bar{\lambda})\|^2 &= \mathcal{O}\Big(\frac{1}{t^2}\Big) \text{ as } t \to +\infty;\\
\|(\dot{x}(t), \dot{\lambda}(t))\|^2 &= \mathcal{O}\Big(\frac{1}{t^2}\Big) \text{ as } t \to +\infty.
\end{aligned}$$

Moreover, if T satisfies condition (L), it further holds that

$$\begin{aligned}
\|(x(t), \lambda(t)) - (\bar{x}, \bar{\lambda})\|^2 &= \mathcal{O}\Big(\frac{1}{t^2}\Big) \text{ as } t \to +\infty;\\
\|(\dot{x}(t), \dot{\lambda}(t))\|^2 &= \mathcal{O}\Big(\frac{1}{t^2} + \frac{1}{t^4}\Big) \text{ as } t \to +\infty.
\end{aligned}$$

With respect to the viscosity curve (x_t, λ_t), we have the following decay rate estimates:

Proposition 4.10. *Let $S \times M$ be non-empty and let $(x, \lambda) : [t_0, +\infty[\to X \times Y$ be a solution of* (AHT). *Let $(x_t, \lambda_t) = \operatorname{zer} T_t$ for each $t \geq t_0$ and let $\varepsilon : [t_0, +\infty[\to]0, +\infty[$ be defined by $\varepsilon(t) = 1/t^p$ with $p \in]0, 1]$ and $t_0 > 0$. Then the following assertions hold:*

$$\begin{aligned}
\|(\dot{x}(t), \dot{\lambda}(t)) + \frac{1}{t^p}((x(t), \lambda(t)) - (x_t, \lambda_t))\|^2 &= \mathcal{O}\Big(\mathrm{e}^{-2\rho(t)} + \frac{1}{t^{2p}}\Big) \textit{ as } t \to +\infty;\\
\|T(x(t), \lambda(t)) - T(x_t, \lambda_t)\|^2 &= \mathcal{O}\Big(\mathrm{e}^{-2\rho(t)} + \frac{1}{t^{2p}}\Big) \textit{ as } t \to +\infty.
\end{aligned}$$

Remark 4.11. In the particular case $\varepsilon(t) = 1/t$, and under the assumption that T satisfies condition (L), we have the refined estimates

$$\begin{aligned}
\|T(x(t), \lambda(t)) - T(x_t, \lambda_t)\|^2 &= \mathcal{O}\Big(\frac{1}{t^2} + \frac{1}{t^4}\Big) \text{ as } t \to +\infty;\\
\|(x(t), \lambda(t)) - (x_t, \lambda_t)\|^2 &= \mathcal{O}\Big(\frac{1}{t^2} + \frac{1}{t^4}\Big) \text{ as } t \to +\infty.
\end{aligned}$$

5. Numerical experiments

In this section, we provide a simple yet representative example that allows for a direct exposition of our main results.

Example. Let $X, Y = \mathbb{R}$ and consider the saddle-value problem

$$\min_{x \in \mathbb{R}} \max_{\lambda \in \mathbb{R}} L(x, \lambda),$$

where $L : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined in terms of the convex-concave and continuously differentiable bifunction $L(x, \lambda) = \lambda(x - 1)$. Let us choose the Tikhonov regularization function $\varepsilon : [t_0, +\infty[\to]0, +\infty[$ as $\varepsilon(t) = 1/t^p$ with $p \in]0, 1]$ and $t_0 > 0$. In this case, the (AHT) differential system reduces to

$$\begin{cases} \dot{x} + \lambda + \dfrac{x}{t^p} = 0 \\ \dot{\lambda} + 1 - x + \dfrac{\lambda}{t^p} = 0. \end{cases}$$

The evolution of the solutions $(x(t), \lambda(t))$ of the (AHT) differential system together with the viscosity curve (x_t, λ_t) as $t \to +\infty$ for different values of the Tikhonov regularization parameter $p \in]0, 1]$ is depicted in Figures 1 and 2. We thereby distinguish the cases $p = 1$ (see Figure 1) and $p \in]0, 1[$ (see Figure 2). In any case, the initial data is set to $(x_0, \lambda_0) = (0, 0)$ and $t_0 = 0.01$.

The particular case $p = 1$. Since $\varepsilon(t)$ tends to zero as $t \to +\infty$ with $\varepsilon \notin \mathcal{L}^1([0, +\infty[)$ and $\dot{\varepsilon} \in \mathcal{L}^1([0, +\infty[)$, we readily observe from Figure 1 that the solution $(x(t), \lambda(t))$ of (AHT) converges, as $t \to +\infty$, to the unique saddle point $(\bar{x}, \bar{\lambda}) = (1, 0)$ of the bifunction L; cf. Proposition 3.7. Moreover, since $\varepsilon(t)$ verifies, for every $t \geq t_0$, the conditions

$$\begin{cases} \varepsilon^2(t) + \dot{\varepsilon}(t) = 0 \\ 2\varepsilon(t)\dot{\varepsilon}(t) + \ddot{\varepsilon}(t) = 0, \end{cases}$$

we obtain, in accordance with Theorem 4.1 and Proposition 4.2, that $\|(\dot{x}(t), \dot{\lambda}(t)) + \varepsilon(t)(x(t), \lambda(t)) - (\bar{x}, \bar{\lambda})\|^2$ and $\|(\dot{x}(t), \dot{\lambda}(t))\|^2$ obey the asymptotic estimate $\mathcal{O}(1/t^2)$ as $t \to +\infty$. In particular, as $\varepsilon(t)$ is decreasing, we have the refined estimate

$$\lim_{t \to +\infty} \frac{1}{t} \int_{t_0}^{t} \tau^2 \|(\dot{x}(\tau), \dot{\lambda}(\tau))\|^2 \, \mathrm{d}\tau < +\infty,$$

which suggests that $\|(\dot{x}(t), \dot{\lambda}(t))\|^2$ vanishes fast as $t \to +\infty$ in the sense of an exponentially weighted moving average; cf. Proposition 4.3. A similar behavior can be observed for $\|(x(t), \lambda(t)) - (x_t, \lambda_t)\|^2$ in Figure 1. Finally, since the operator $T : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ associated with L, given by $T(x, \lambda) = (\lambda, 1 - x)$, verifies condition (L) with $\alpha = 1$, we readily observe that $\|(x(t), \lambda(t)) - (\bar{x}, \bar{\lambda})\|^2$ obeys the asymptotic estimate $\mathcal{O}(1/t^2)$ as $t \to +\infty$; see Corollary 4.4. In this scenario, we also find that $\|(\dot{x}_t, \dot{\lambda}_t)\|^2$ vanishes, as predicted by Lemma 2.7, according to the fast asymptotic estimate $\mathcal{O}(1/t^4)$ as $t \to +\infty$.

The particular case $p \in]0, 1[$. Analyzing Figure 2, we observe that the solutions $(x(t), \lambda(t))$ of the (AHT) differential system still admit favorable convergence properties, but their decay rate is considerably degraded as the value of the Tikhonov regularization parameter p decreases. As predicted by Theorem 4.1 and Proposition 4.2, we find that $\|(\dot{x}(t), \dot{\lambda}(t)) + \varepsilon(t)(x(t), \lambda(t)) - (\bar{x}, \bar{\lambda})\|^2$ and $\|(\dot{x}(t), \dot{\lambda}(t))\|^2$ obey the asymptotic estimate $\mathcal{O}\big(\mathrm{e}^{-2\rho(t)} + 1/t^{2p}\big)$ as $t \to +\infty$. However, this estimate is no longer sharp due to the conservatism introduced by the inequalities

$$\left.\begin{aligned} \varepsilon^2(t) + \dot{\varepsilon}(t) \geq 0 \\ 2\varepsilon(t)\dot{\varepsilon}(t) + \ddot{\varepsilon}(t) \leq 0 \end{aligned}\right\} \quad \forall t \geq t_+$$

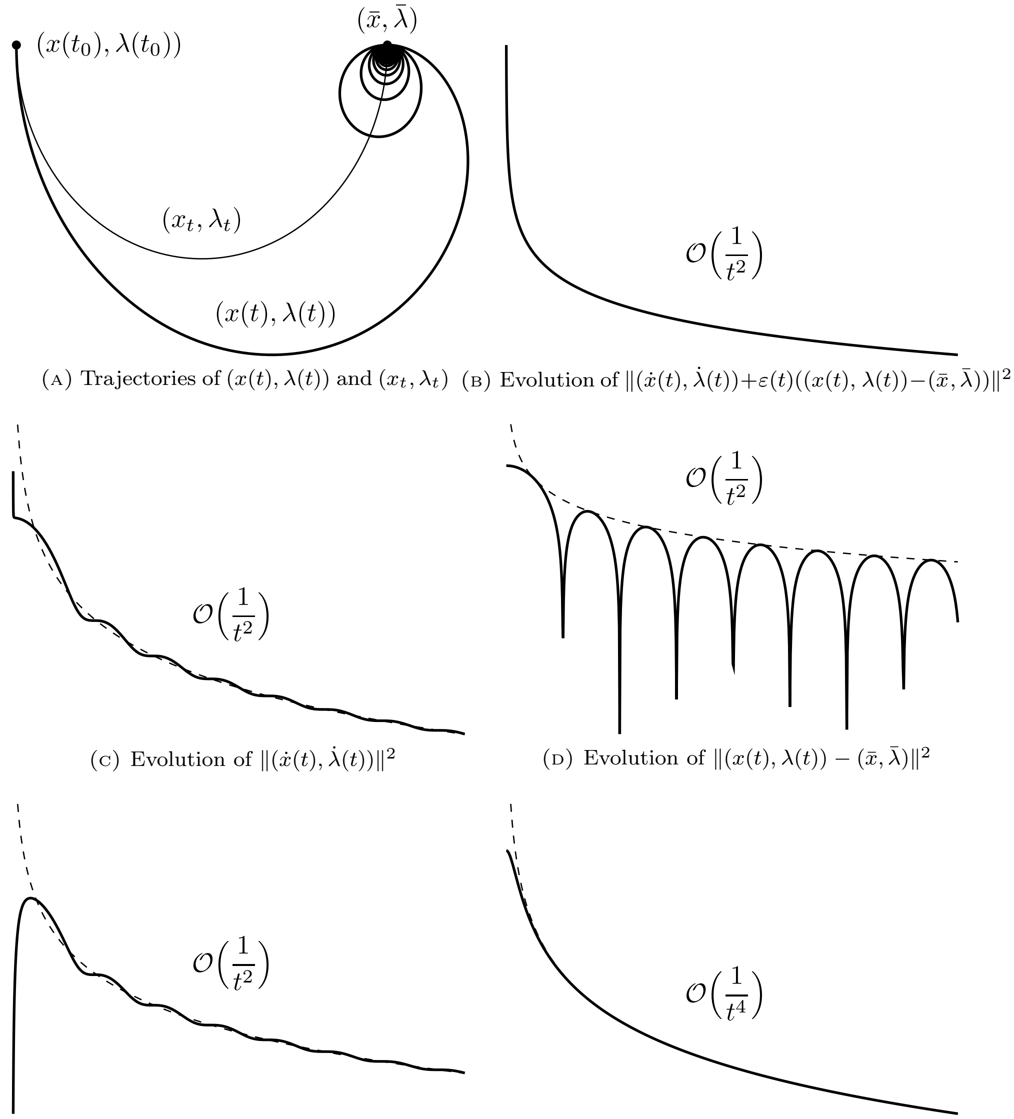
for some $t_+ \geq t_0$. Given this observation, we recover the fact that the fastest convergence rate estimates are obtained whenever the Tikhonov regularization function $\varepsilon : [t_0, +\infty[\to]0, +\infty[$ is chosen according to the differential equation

$$\dot{\varepsilon}(t) + \varepsilon^2(t) = 0,$$

whose solutions are given in terms of

$$\varepsilon(t) = \frac{1}{t + c}$$

for some constant $c \geq 0$. We leave the discussion on sharp asymptotic decay rate estimates for the solutions $(x(t), \lambda(t))$ of (AHT) in the case when the Tikhonov regularization parameter p is chosen in $]0, 1[$ open for future investigations.



(A) Trajectories of $(x(t), \lambda(t))$ and (x_t, λ_t) (B) Evolution of $\|(\dot{x}(t), \dot{\lambda}(t)) + \varepsilon(t)((x(t), \lambda(t)) - (\bar{x}, \bar{\lambda}))\|^2$

(C) Evolution of $\|(\dot{x}(t), \dot{\lambda}(t))\|^2$ (D) Evolution of $\|(x(t), \lambda(t)) - (\bar{x}, \bar{\lambda})\|^2$

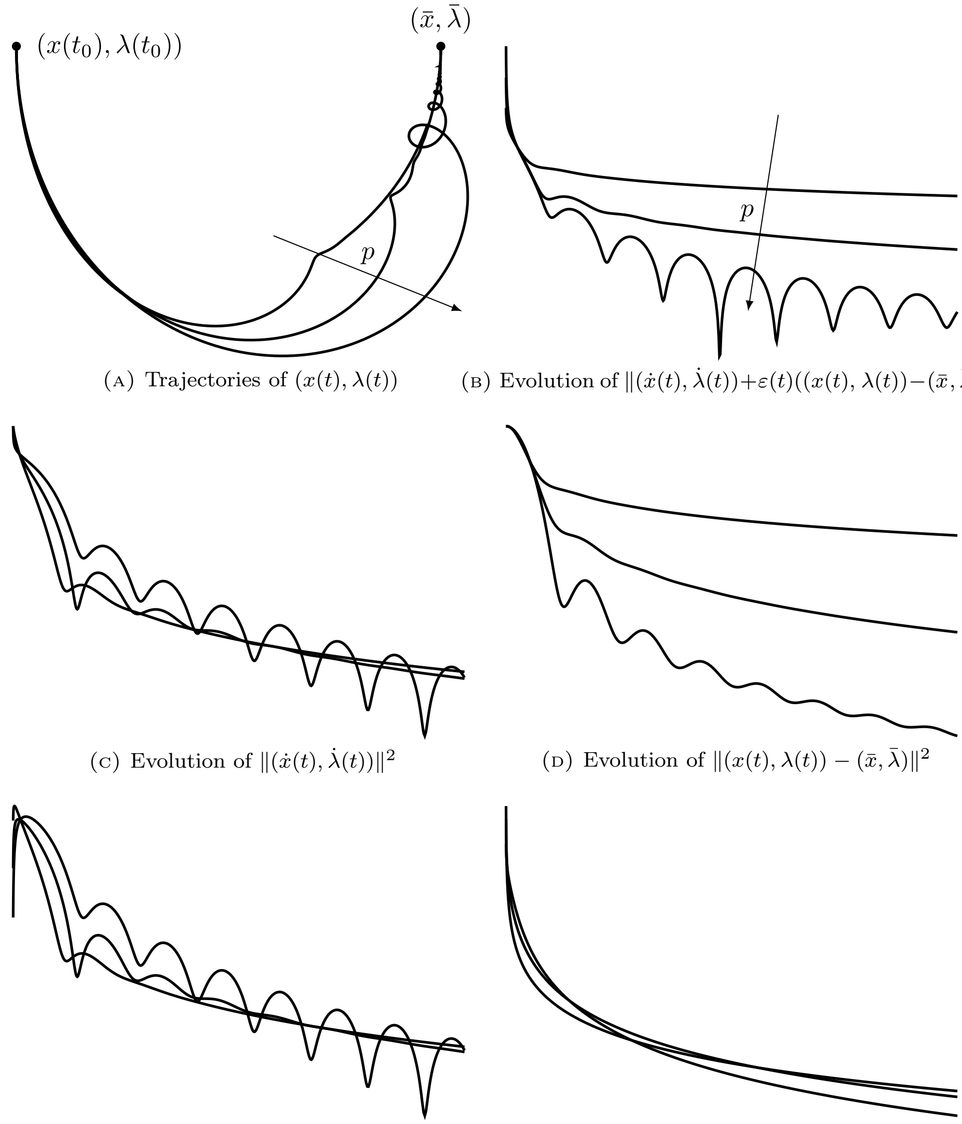
(E) Evolution of $\|(x(t), \lambda(t)) - (x_t, \lambda_t)\|^2$ (F) Evolution of $\|(\dot{x}_t, \dot{\lambda}_t)\|^2$

FIGURE 1. Graphical view on the evolution of a solution $(x(t), \lambda(t))$ of the (AHT) differential system together with the viscosity curve (x_t, λ_t) as $t \to +\infty$ for the Tikhonov regularization parameter $p = 1$.

APPENDIX

We collect here some auxiliary results which are used in the asymptotic analysis of the solutions of the (AHT) differential system.

Let us first recall the following classical result as outlined in Cominetti et al. [17, Lemma 1].



(A) Trajectories of $(x(t), \lambda(t))$ (B) Evolution of $\|(\dot{x}(t), \dot{\lambda}(t))+\varepsilon(t)((x(t), \lambda(t))-(\bar{x}, \bar{\lambda}))\|^2$

(C) Evolution of $\|(\dot{x}(t), \dot{\lambda}(t))\|^2$ (D) Evolution of $\|(x(t), \lambda(t)) - (\bar{x}, \bar{\lambda})\|^2$

(E) Evolution of $\|(x(t), \lambda(t)) - (x_t, \lambda_t)\|^2$ (F) Evolution of $\|(\dot{x}_t, \dot{\lambda}_t)\|^2$

FIGURE 2. Graphical view on the evolution of the (AHT) solutions $(x(t), \lambda(t))$ together with the viscosity curve (x_t, λ_t) for the Tikhonov regularization parameters $p = 1/4$, $p = 1/2$, and $p = 3/4$.

Lemma A.1. *Let $\phi : [t_0, +\infty[\to \mathbb{R}$ be continuously differentiable, let $\vartheta : [t_0, +\infty[\to \mathbb{R}$ be bounded, and let $\varepsilon : [t_0, +\infty[\to [0, +\infty[$ be locally integrable such that*

$$\dot{\phi}(t) + \varepsilon(t)\phi(t) \leq \varepsilon(t)\vartheta(t) \quad \forall t \geq t_0.$$

Then $\phi(t)$ remains bounded on $[t_0, +\infty[$. Moreover, if $\varepsilon \notin \mathcal{L}^1([t_0, +\infty[)$, then

$$\limsup_{t \to +\infty} \phi(t) \leq \limsup_{t \to +\infty} \vartheta(t).$$

For the following basic inequality of Gronwall-type, the reader is referred to Brézis [14, Lemma A.5].

Lemma A.2. *Let* $\phi : [t_0, +\infty[\to \mathbb{R}$ *be continuous and non-negative, and let* $\vartheta : [t_0, +\infty[\to [0, +\infty[$ *be locally integrable such that*

$$\frac{1}{2}\phi^2(t) \leq \frac{1}{2}\phi^2(t_0) + \int_{t_0}^{t} \vartheta(\tau)\phi(\tau)\,\mathrm{d}\tau \quad \forall t \geq t_0.$$

Then it holds that

$$\phi(t) \leq \phi(t_0) + \int_{t_0}^{t} \vartheta(\tau)\,\mathrm{d}\tau \quad \forall t \geq t_0.$$

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