Ergodic convergence results for the Arrow–Hurwicz differential system

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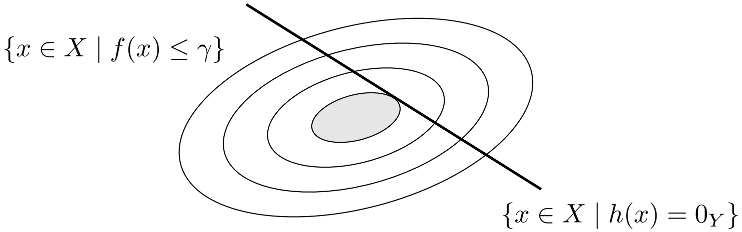
Problem statement

Let X, Y be real Hilbert spaces endowed with inner products $\langle \cdot\,, \cdot \rangle_X$, $\langle \cdot\,, \cdot \rangle_Y$ and induced norms $\| \cdot \|_X$, $\| \cdot \|_Y$.

Problem. Consider the minimization problem

$$\text{minimize } f(x) \quad \text{subject to } h(x) = 0_Y. \tag{P}$$

- $f : X \to \mathbb{R}$ is convex and continuously differentiable
- $h : X \to Y$ is continuous and affine

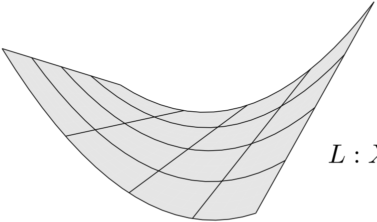


The Arrow–Hurwicz differential system

Arrow–Hurwicz differential system. We reconsider the classical first-order evolution system[1]

$$\begin{cases} \dot{x} + \nabla f(x) + h'(x)^* \lambda = 0_X \\ \dot{\lambda} - h(x) = 0_Y \end{cases} \tag{AH}$$

in view of solving the convex minimization problem (P).



$$\begin{aligned} L : X \times Y &\longrightarrow \mathbb{R} \\ (x, \lambda) &\longmapsto f(x) + \langle \lambda, h(x) \rangle_Y \end{aligned}$$

[1] K. J. Arrow and L. Hurwicz, *A gradient method for approximating saddle points and constrained maxima*, RAND Corp., Santa Monica, CA, pp. p-223, 1951.

Outline

Introduction

Basic properties
(Maximal) monotonicity, integrability estimate, ...

Weak ergodic convergence
Limiting average behavior, localization of the weak limit, ...

Refined ergodic estimates
"Primal-dual gap function", refined asymptotics, ...

Further extension
Liénard-type inertial dynamics, ...

Conclusions

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Preliminaries

Let us associate with (P) the *Lagrangian*

$$
\begin{aligned}
L : X \times Y &\longrightarrow \mathbb{R} \\
(x, \lambda) &\longmapsto f(x) + \langle \lambda, h(x) \rangle_Y .
\end{aligned}
$$

Definition. A pair $(\bar{x}, \bar{\lambda}) \in X \times Y$ is a *saddle point* of L if

$$
L(\bar{x}, \lambda) \leq L(\bar{x}, \bar{\lambda}) \leq L(x, \bar{\lambda}) \quad \forall (x, \lambda) \in X \times Y.
$$

We denote by $S \times M \subset X \times Y$ the set of saddle points of L.

Assumptions.

- $f : X \to \mathbb{R}$ is convex and continuously differentiable
- $\nabla f : X \to X$ is Lipschitz continuous on bounded sets
- $A : X \to Y$ is linear and continuous, $b \in Y$, and

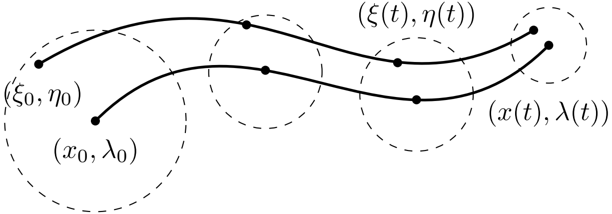
$$
\begin{aligned}
h : X &\longrightarrow Y \\
x &\longmapsto Ax - b
\end{aligned}
$$

(Maximal) monotonicity

Given our basic assumptions, we have the following important property concerning the (AH) differential system:

Main feature. *(Maximal) monotonicity* of the "(AH) generator"[2]

$$\begin{aligned} T : X \times Y &\longrightarrow X \times Y \\ (x, \lambda) &\longmapsto (\nabla f(x) + A^* \lambda, b - Ax). \end{aligned}$$



[2] R. T. Rockafellar, *Monotone operators associated with saddle-functions and minimax problems*, in Nonlinear Functional Analysis, Amer. Math. Soc., pp. 241-250, 1969.

Integrability estimate

Consider the "primal-dual gap function" (relative to $S \times M$)

$$t \longmapsto L(x(t), \cdot\,) - L(\,\cdot\,, \lambda(t))$$

as a natural measure of optimality.

Proposition. Let $S \times M$ be non-empty and let $(x, \lambda) : [0, +\infty) \to X \times Y$ be a solution of (AH). Then, for any $(\xi, \eta) \in S \times M$, it holds that

$$\int_0^\infty L(x(\tau), \eta) - L(\xi, \lambda(\tau))\, \mathrm{d}\tau < +\infty.$$

Define the *Cesàro average* of a solution (x, λ) of (AH) as

$$\begin{aligned}(\sigma, \omega) : (0, +\infty) &\longrightarrow X \times Y \\ t &\longmapsto \frac{1}{t}\int_0^t (x(\tau), \lambda(\tau))\, \mathrm{d}\tau\,.\end{aligned}$$

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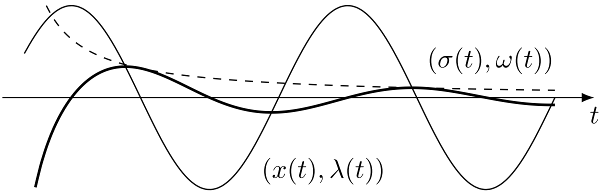
Weak ergodic convergence

Theorem. Let $S \times M$ be non-empty and let $(\sigma, \omega) : (0, +\infty) \to X \times Y$ be the Cesàro average of a solution of (AH). Then, for any $(\xi, \eta) \in S \times M$, it holds that

$$L(\sigma(t), \eta) - L(\xi, \omega(t)) = \mathcal{O}\Big(\frac{1}{t}\Big) \text{ as } t \to +\infty.$$

Moreover, there exists $(\bar{\sigma}, \bar{\omega}) \in S \times M$ such that $(\sigma(t), \omega(t)) \rightharpoonup (\bar{\sigma}, \bar{\omega})$ weakly in $X \times Y$ as $t \to +\infty$.

Corollary. If $S \times M$ is empty, then $\lim_{t \to +\infty} \|(\sigma(t), \omega(t))\| = +\infty$.



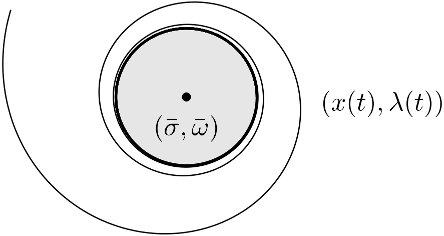
Localization of the weak limit

Given a *bounded* solution (x, λ) of (AH), consider[3]

$$\phi(\xi, \eta) = \limsup_{t \to +\infty} \|(x(t), \lambda(t)) - (\xi, \eta)\|^2.$$

Proposition. Let $S \times M$ be non-empty and let $(\bar{\sigma}, \bar{\omega}) \in S \times M$ be such that $(\sigma(t), \omega(t)) \rightharpoonup (\bar{\sigma}, \bar{\omega})$ weakly in $X \times Y$ as $t \to +\infty$. Then,

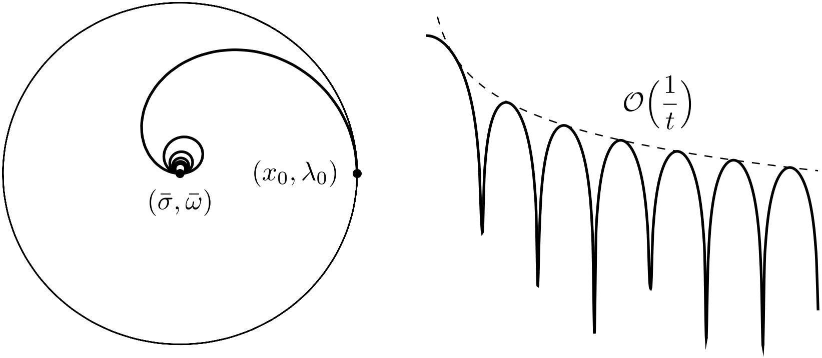
$$\phi(\bar{\sigma}, \bar{\omega}) \leq \phi(\xi, \eta) \quad \forall (\xi, \eta) \in X \times Y.$$



[3] M. Edelstein, *The construction of an asymptotic center with a fixed-point property*, Bull. Amer. Math. Soc., 78:206-208, 1972.

Numerical experiment

Illustration.



$\{(\sigma(t), \omega(t)) \mid t > 0\}$ $\qquad$ $\|(\sigma(t), \omega(t)) - (\bar{\sigma}, \bar{\omega})\|$

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Refined ergodic estimates

Let us assume that $A : X \to Y$ is bounded from below, i.e.,

$$\exists \beta > 0 \; \forall x \in X, \quad \|Ax\|_Y \geq \beta \|x\|_X .$$

Proposition. Let $S \times M$ be non-empty, let $A : X \to Y$ be bounded from below, and let $(\sigma, \omega) : (0, +\infty) \to X \times Y$ be the Cesàro average of a solution of (AH). Then, for any $(\xi, \eta) \in S \times M$, it holds that

$$L(\sigma(t), \eta) - L(\xi, \omega(t)) = \mathcal{O}\Big(\frac{1}{t^2}\Big) \text{ as } t \to +\infty;$$

$$\|\sigma(t) - \xi\|_X = \mathcal{O}\Big(\frac{1}{t}\Big) \text{ as } t \to +\infty.$$

Implication.

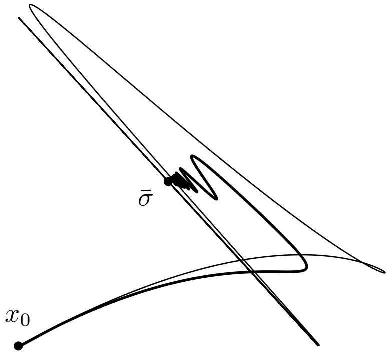
$S \times M = \{ \bullet \} \times \{$ $\}$

$\bullet$ . . . unique minimizer of (P)

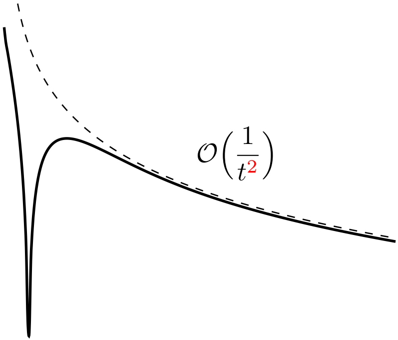
 . . . affine subspace of Lagrange multipliers

Numerical experiment

Illustration.



$\{\sigma(t) \mid t > 0\}$



$L(\sigma(t), \bar{\omega}) - L(\bar{\sigma}, \omega(t))$

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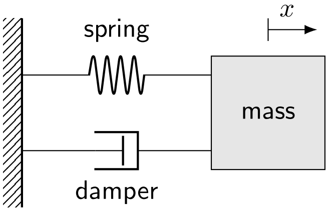
A Liénard-type differential system

The Arrow–Hurwicz differential system (AH) admits an equivalent second-order representation in terms of the

Liénard-type inertial dynamics. Consider the second-order evolution system[4]

$$\ddot{x} + \nabla^2 f(x)\dot{x} + \nabla\|h(x)\|_Y^2/2 = 0_X \qquad \text{(ID)}$$

relative to the convex minimization problem (P).



[4] A. Liénard, *Étude des oscillations entretenues*, Rev. gén. d'électr., 23:901-912 and 946-954, 1928.

Link between the dynamics

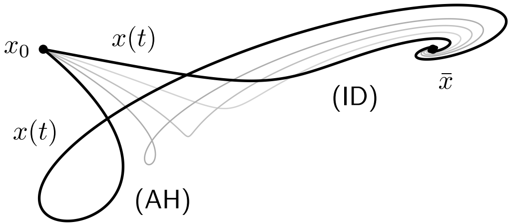
We have the following relation between the Arrow–Hurwicz differential system (AH) and the Liénard-type inertial dynamics (ID):

Connection.

$f \in \mathcal{C}^2$ (→)

$$\begin{cases} \dot{x} + \nabla f(x) + h'(x)^* \lambda = 0_X \\ \dot{\lambda} - h(x) = 0_Y \end{cases} \qquad \ddot{x} + \nabla^2 f(x)\dot{x} + \nabla \|h(x)\|_Y^2/2 = 0_X$$

(←) $h'(\,\cdot\,)$ surjective



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Thank you for your attention!

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