Asymptotic behavior of the nonautonomous Arrow–Hurwicz differential system

Simon K. Niederländer

Autonomous Systems and Control
Siemens Foundational Technologies
Munich, Germany

63rd IEEE Conference on Decision and Control
Milan, Italy, December 18, 2024

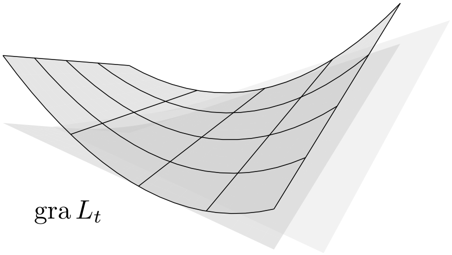
Problem statement

Let X, Y be real Hilbert spaces endowed with inner products $\langle \cdot\,, \cdot \rangle_X$, $\langle \cdot\,, \cdot \rangle_Y$ and associated norms $\| \cdot \|_X$, $\| \cdot \|_Y$.

Problem. Consider the saddle-value problem

$$\inf_{x \in X} \sup_{\lambda \in Y} L_t(x, \lambda), \qquad (\mathsf{P}_t)$$

where for each $t \geq 0$, $L_t : X \times Y \to \mathbb{R}$ is a convex-concave and continuously differentiable bifunction.



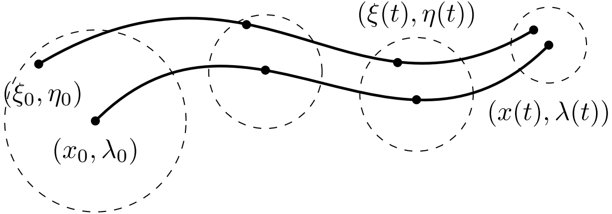
The Arrow–Hurwicz differential system

Arrow–Hurwicz differential system. We consider the nonautonomous evolution system[1]

$$\begin{cases} \dot{x} + \nabla_x L_t(x,\lambda) = 0 \\ \dot{\lambda} - \nabla_\lambda L_t(x,\lambda) = 0 \end{cases} \qquad \text{(NAH)}$$

relative to the saddle-value problem (P_t).

We say that $(x,\lambda) : [0,+\infty[\, \to X \times Y$ is a (classical) solution of (NAH) if $(x,\lambda) \in \mathcal{C}^1([0,+\infty[)$ such that (NAH) is satisfied on $[0,+\infty[$.



[1] K. J. Arrow and L. Hurwicz, *A gradient method for approximating saddle points and constrained maxima*, RAND Corp., Santa Monica, CA, pp. p-223, 1951.

Outline

Introduction

Preliminary results
Integrability estimate, "no-regret condition", ...

Weak ergodic convergence
Gap function, asymptotic average, ...

Weak convergence
Strict convexity-concavity, ...

Application and extension
Tikhonov regularization, ...

Conclusions

Outline

Introduction

Preliminary results

Integrability estimate, “no-regret condition”, ...

Weak ergodic convergence

Gap function, asymptotic average, ...

Weak convergence

Strict convexity-concavity, ...

Application and extension

Tikhonov regularization, ...

Conclusions

Preliminaries

Proposition. Let $(x, \lambda) : [0, +\infty[\to X \times Y$ be a solution of (NAH). Then, for every $(\xi, \eta) \in X \times Y$, it holds that

$$\limsup_{t \to +\infty} \int_0^t L_\tau(x(\tau), \eta) - L_\tau(\xi, \lambda(\tau)) \, \mathrm{d}\tau < +\infty.$$

Remark ("No-regret condition"). For every $t \geq 0$, let the "regret function" $\mathrm{Regret}_t : X \times Y \to \mathbb{R}$ be defined by

$$\mathrm{Regret}_t(\xi, \eta) = \int_0^t L_\tau(x(\tau), \eta) - L_\tau(\xi, \lambda(\tau)) \, \mathrm{d}\tau \, .$$

Then, for every $(\xi, \eta) \in X \times Y$, we have

$$\mathrm{Regret}_t(\xi, \eta) \leq \mathcal{O}(t) \text{ as } t \to +\infty.$$

Preparatory result

Proposition. Let $(x, \lambda) : [0, +\infty[\to X \times Y$ be a solution of (NAH) and suppose that there exists a closed convex-concave bifunction[2] $L_\infty : X \times Y \to \mathbb{R}$ such that for every $(\xi, \eta) \in X \times Y$,

$$L_t(\,\cdot\,, \eta) - L_t(\xi, \,\cdot\,) \to L_\infty(\,\cdot\,, \eta) - L_\infty(\xi, \,\cdot\,)$$

uniformly on $X \times Y$ as $t \to +\infty$.

If $(\bar{x}, \bar{\lambda}) \in X \times Y$ is such that $(x(t), \lambda(t)) \to (\bar{x}, \bar{\lambda})$ strongly in $X \times Y$ as $t \to +\infty$, then, for every $(\xi, \eta) \in X \times Y$,

$$L_\infty(\bar{x}, \eta) \le L_\infty(\bar{x}, \bar{\lambda}) \le L_\infty(\xi, \bar{\lambda}).$$

Remark. If L_t tends to L_∞ (in the above sense) as $t \to +\infty$, then the limit of a solution of (NAH) is necessarily a saddle point of L_∞.

[2] For each $(\xi, \eta) \in X \times Y$, the functions $L_\infty(\,\cdot\,, \eta)$ and $-L_\infty(\xi, \,\cdot\,)$ are convex and lower semicontinuous.

Outline

Introduction

Preliminary results
Integrability estimate, “no-regret condition”, ...

Weak ergodic convergence
Gap function, asymptotic average, ...

Weak convergence
Strict convexity-concavity, ...

Application and extension
Tikhonov regularization, ...

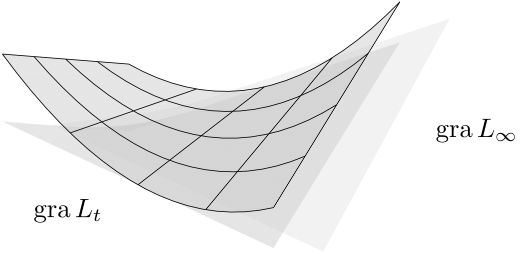
Conclusions

Gap function

Assumption. Suppose there exists a closed convex-concave bifunction $L_\infty : X \times Y \to \mathbb{R}$ with a non-empty set of saddle points $S \times M$ such that the gap function $\mathrm{GAP}_{L_t - L_\infty} : X \times Y \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\mathrm{GAP}_{L_t - L_\infty}(\xi, \eta) = \sup_{\mu \in Y} \big(L_t(\xi, \mu) - L_\infty(\xi, \mu) \big) - \inf_{\nu \in X} \big(L_t(\nu, \eta) - L_\infty(\nu, \eta) \big)$$

vanishes "sufficiently fast" as $t \to +\infty$.



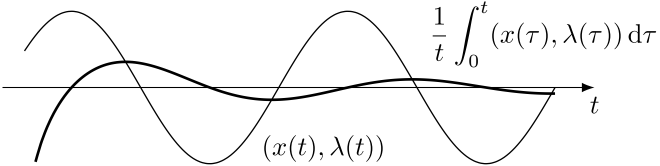
Weak ergodic convergence

Theorem. Let $(x,\lambda) : [0,+\infty[\to X \times Y$ be a solution of (NAH) and suppose that there exists a closed convex-concave bifunction $L_\infty : X \times Y \to \mathbb{R}$ with a non-empty set of saddle points $S \times M$ such that

$$\forall(\xi,\eta) \in X \times Y, \quad \int_0^\infty \mathrm{GAP}_{L_\tau - L_\infty}(\xi,\eta)\,\mathrm{d}\tau < +\infty.$$

Then there exists $(\bar{x},\bar{\lambda}) \in S \times M$ such that

$$\mathrm{w} - \lim_{t\to+\infty} \frac{1}{t}\int_0^t (x(\tau),\lambda(\tau))\,\mathrm{d}\tau = (\bar{x},\bar{\lambda}).$$



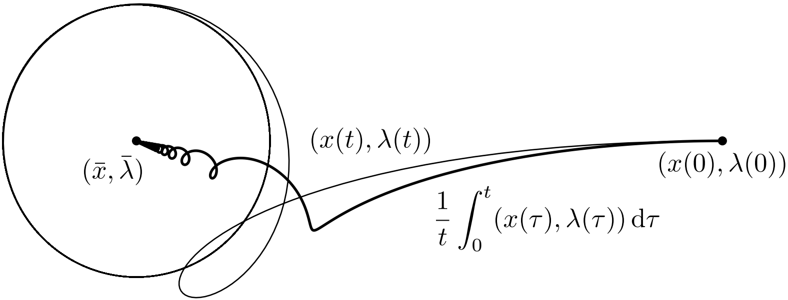
Numerical experiment

Example. Let $X, Y = \mathbb{R}$ and consider, for every $t \geq 0$,

$$L_t(x,\lambda) = \frac{\mathrm{e}^{-t}}{2}(x^2 - y^2) + (\lambda - 1)(x-1),$$

so that $L_\infty(x,\lambda) = (\lambda-1)(x-1)$ with $S \times M = \{(1,1)\}$ and $\mathrm{GAP}_{L_t - L_\infty}(\xi,\eta) = \mathrm{e}^{-t}\|(\xi,\eta)\|^2/2$.

Illustration.



Outline

Introduction

Preliminary results
Integrability estimate, "no-regret condition", ...

Weak ergodic convergence
Gap function, asymptotic average, ...

Weak convergence
Strict convexity-concavity, ...

Application and extension
Tikhonov regularization, ...

Conclusions

Weak convergence

Theorem. Let $(x, \lambda) : [0, +\infty[\to X \times Y$ be a solution of (NAH) and suppose that there exists a closed convex-concave bifunction $L_\infty : X \times Y \to \mathbb{R}$ with a non-empty set of saddle points $S \times M$ such that

$$\forall (\xi, \eta) \in X \times Y, \quad \int_0^\infty \mathrm{GAP}_{L_\tau - L_\infty}(\xi, \eta)\,\mathrm{d}\tau < +\infty.$$

Assume, in addition, that the bifunction L_∞ is such that for all $(\bar{x}, \bar{\lambda}) \in S \times M$ and $(\xi, \eta) \notin S \times M$,[3]

$$L_\infty(\bar{x}, \eta) < L_\infty(\bar{x}, \bar{\lambda}) < L_\infty(\xi, \bar{\lambda}).$$

Then there exists $(\bar{x}, \bar{\lambda}) \in S \times M$ such that

$$\mathrm{w} - \lim_{t \to +\infty} (x(t), \lambda(t)) = (\bar{x}, \bar{\lambda}).$$

[3] R. T. Rockafellar, *Saddle-points and convex analysis*, in Differential Games and Related Topics, North-Holland, pp. 109-127, 1971.

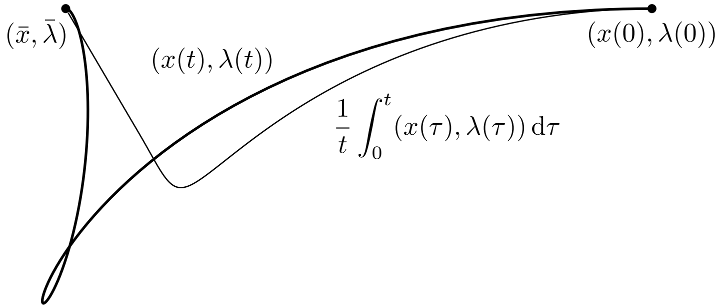
Numerical experiment

Example. Let $X, Y = \mathbb{R}$ and consider now, for every $t \geq 0$,

$$K_t(x, \lambda) = L_t(x, \lambda) + \frac{1}{2}(x-1)^2 - \frac{1}{2}(\lambda - 1)^2,$$

so that $K_\infty(x, \lambda) = L_\infty + (x-1)^2/2 - (\lambda-1)^2/2$, $S \times M = \{(1,1)\}$, and $\mathrm{GAP}_{K_t - K_\infty}(\xi, \eta) = \mathrm{GAP}_{L_t - L_\infty}(\xi, \eta)$.

Illustration.



Outline

Introduction

Preliminary results
Integrability estimate, “no-regret condition”, . . .

Weak ergodic convergence
Gap function, asymptotic average, . . .

Weak convergence
Strict convexity-concavity, . . .

Application and extension
Tikhonov regularization, . . .

Conclusions

Tikhonov regularization

Let $\varepsilon : [0, +\infty[\to]0, +\infty[$ be continuously differentiable such that

$$\lim_{t \to +\infty} \varepsilon(t) = 0.$$

Arrow–Hurwicz differential system with Tikhonov regularization. Consider the nonautonomous evolution system

$$\begin{cases} \dot{x} + \nabla_x L(x, \lambda) + \varepsilon(t) x = 0 \\ \dot{\lambda} - \nabla_\lambda L(x, \lambda) + \varepsilon(t) \lambda = 0 \end{cases} \tag{AHT}$$

in view of solving the saddle-value problem (P_t).

This amounts to the mini-maximization of

$$\begin{aligned} L_t : X \times Y &\longrightarrow \mathbb{R} \\ (x, \lambda) &\longmapsto L(x, \lambda) + \frac{\varepsilon(t)}{2} \left(\|x\|_X^2 - \|\lambda\|_Y^2 \right), \end{aligned}$$

where $L : X \times Y \to \mathbb{R}$ is a convex-concave and continuously differentiable bifunction (with a non-empty set of saddle points $S \times M$).

Associated gap function

The gap function $\mathrm{GAP}_{L_t - L} : X \times Y \to \mathbb{R} \cup \{+\infty\}$ reduces to

$$\mathrm{GAP}_{L_t-L}(\xi,\eta) = \sup_{\mu \in Y} \big(L_t(\xi,\mu) - L(\xi,\mu)\big) - \inf_{\nu \in X} \big(L_t(\nu,\eta) - L(\nu,\eta)\big) = \frac{\varepsilon(t)}{2}\|(\xi,\eta)\|^2.$$

Corollary. Let $S \times M$ be non-empty, let $(x,\lambda) : [0,+\infty[\ \to X \times Y$ be a solution of (AHT), and suppose that $\varepsilon \in \mathcal{L}^1([0,+\infty[)$. Then there exists $(\bar{x},\bar{\lambda}) \in S \times M$ such that

$$\mathrm{w} - \lim_{t \to +\infty} \frac{1}{t} \int_0^t (x(\tau),\lambda(\tau))\,\mathrm{d}\tau = (\bar{x},\bar{\lambda}).$$

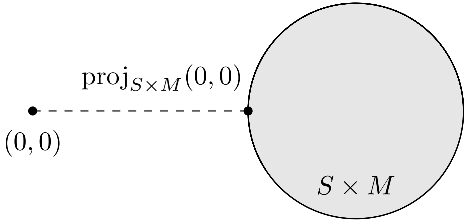
If, moreover, L is "strictly convex-concave", then

$$\mathrm{w} - \lim_{t \to +\infty} (x(t),\lambda(t)) = (\bar{x},\bar{\lambda}).$$

The particular case $\varepsilon \notin \mathcal{L}^1([0,+\infty[)$

Proposition. Let $S \times M$ be non-empty, let $(x,\lambda) : [0,+\infty[\to X \times Y$ be a solution of (AHT), and suppose that $\varepsilon \notin \mathcal{L}^1([0,+\infty[)$ with either $\dot{\varepsilon} \in \mathcal{L}^1([0,+\infty[)$ or $|\dot{\varepsilon}|^2/\varepsilon \in \mathcal{L}^1([0,+\infty[)$. Then it holds that[4]

$$\lim_{t\to+\infty}(x(t),\lambda(t)) = \operatorname{proj}_{S\times M}(0,0).$$



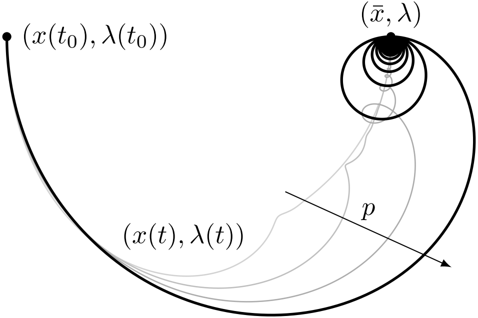
[4] F. Battahi, Z. Chbani, S. K. Niederländer, and H. Riahi, *Asymptotic behavior of the Arrow–Hurwicz differential system with Tikhonov regularization*, (2024), available at https://arxiv.org/abs/2411.17656.

Numerical experiment

Example. Let $X, Y = \mathbb{R}$, take $L(x,\lambda) = \lambda(x-1)$, and consider the Tikhonov regularization function $\varepsilon(t) = 1/t^p$ with $p \in]0,1]$ and $t_0 > 0$. The (AHT) differential system reduces to

$$\begin{cases} \dot{x} + \lambda + \dfrac{x}{t^p} = 0 \\ \dot{\lambda} + 1 - x + \dfrac{\lambda}{t^p} = 0. \end{cases}$$

Illustration.



Outline

Introduction

Preliminary results
Integrability estimate, "no-regret condition", ...

Weak ergodic convergence
Gap function, asymptotic average, ...

Weak convergence
Strict convexity-concavity, ...

Application and extension
Tikhonov regularization, ...

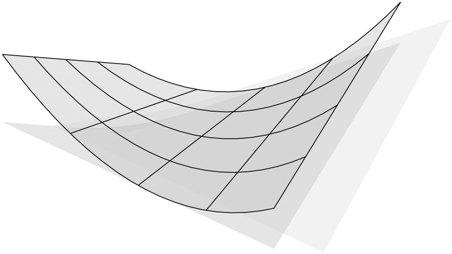
Conclusions

Conclusions

Case I. If the "nonautonomous part" of the (NAH) differential system vanishes "sufficiently fast" as $t \to +\infty$, then the asymptotic behavior is characterized by the "autonomous part":

(i) In general only weak ergodic convergence;

(ii) If "limiting saddle function" is strict, then weak convergence.



Case II. If the "nonautonomous part" of the (NAH) differential system vanishes "sufficiently slow" as $t \to +\infty$, then it asymptotically dominates the "autonomous part".

Thank you for your attention!

CDC·2024 MILANO