The continuous steepest descent method with convex-like potential

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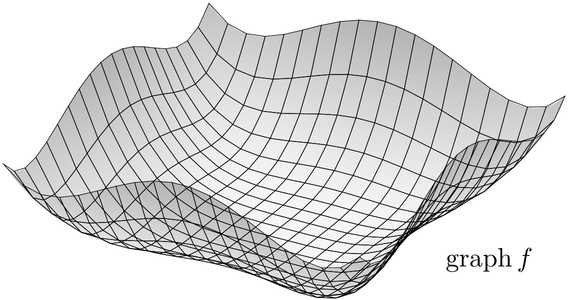
Problem statement

Let $\mathcal{H}$ be a real Hilbert space endowed with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$.

Problem. Consider the minimization problem

$$\inf \{f(x) : x \in \mathcal{H}\}, \tag{P}$$

where $f : \mathcal{H} \to \mathbb{R}$ is a continuously differentiable real-valued function.



Continuous steepest descent

Assumptions.

- $f : \mathcal{H} \to \mathbb{R}$ is minorized, i.e., $\inf_{\mathcal{H}} f > -\infty$;
- $\nabla f : \mathcal{H} \to \mathcal{H}$ is Lipschitz continuous on bounded sets.

Continuous steepest descent. Consider the first-order evolution equation

$$\dot{x} + \nabla f(x) = 0 \tag{SD}$$

relative to the minimization problem (P).

Preliminary facts.

(i) $t \mapsto f(x(t))$ is non-increasing;

(ii) "Finite-energy property":

$$\int_0^\infty \|\dot{x}(\tau)\|^2 \, \mathrm{d}\tau < +\infty;$$

(iii) $x([0, +\infty[)$ bounded $\implies \lim_{t\to+\infty} \dot{x}(t) = \lim_{t\to+\infty} \nabla f(x(t)) = 0.$

Convex case

Selected results. $f : \mathcal{H} \to \mathbb{R}$ convex and $\operatorname{argmin}_{\mathcal{H}} f \neq \emptyset$

(i) **Minimizing properties** (Brézis, 1971):

$$\lim_{t \to +\infty} f(x(t)) = \min\nolimits_{\mathcal{H}} f;$$
$$f(x(t)) - \min\nolimits_{\mathcal{H}} f = \mathcal{O}\left(\frac{1}{t}\right) \text{ as } t \to +\infty.$$

(ii) **Weak convergence** (Bruck, 1975): $\exists \bar{x} \in \operatorname{argmin}_{\mathcal{H}} f$ such that

$$\mathrm{w}-\lim_{t \to +\infty} x(t) = \bar{x}.$$

(iii) **Asymptotic estimates** (Attouch–Boț–Nguyễn, 2024):

$$\int_0^\infty \tau \|\dot{x}(\tau)\|^2 \, \mathrm{d}\tau < +\infty;$$
$$f(x(t)) - \min\nolimits_{\mathcal{H}} f = o\left(\frac{1}{t}\right) \text{ as } t \to +\infty.$$

Convex-like condition

Convex-like condition. Let $f \in \mathcal{C}^1(\mathcal{H};\mathbb{R})$ satisfy $\operatorname{argmin}_{\mathcal{H}} f \neq \emptyset$ and suppose that there exists $\theta > 0$ such that, for every $x \in \mathcal{H}$ and every $\bar{x} \in \operatorname{argmin}_{\mathcal{H}} f$,

$$\langle \nabla f(x), x - \bar{x} \rangle \geq \theta\big(f(x) - \min_{\mathcal{H}} f\big). \tag{C}$$

Sufficient structures.

(i) **Convex power of the objective gap** (Su–Boyd–Candès, 2016):

$$x \longmapsto \big(f(x) - \min_{\mathcal{H}} f\big)^{1/\theta} \text{ convex}$$
$$\implies \text{Condition (C);}$$

(ii) **Positive homogeneity/Euler** (Cabot–Engler–Gadat, 2009):

$$f(\lambda x) = \lambda^\theta f(x),\ \operatorname{argmin}_{\mathcal{H}} f = \{0\} \implies \langle \nabla f(x), x \rangle = \theta f(x)$$
$$\implies \text{Condition (C).}$$

Every critical point of f is a minimizer, i.e., $\operatorname{crit} f \subset \operatorname{argmin}_{\mathcal{H}} f$.

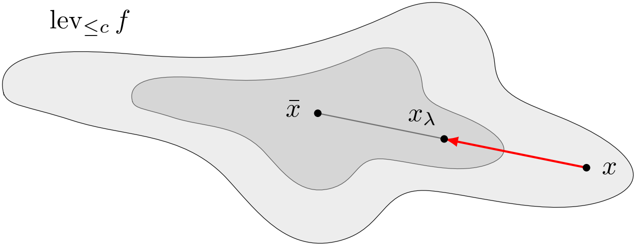
Geometric interpretation

For $x \in \mathcal{H}$, $\bar{x} \in \operatorname{argmin}_{\mathcal{H}} f$, and $\lambda \in [0,1]$, set

$$x_\lambda := \bar{x} + \lambda(x - \bar{x}).$$

Condition (C) yields the "**radial contraction estimate**"

$$f(x_\lambda) - \min_{\mathcal{H}} f \leq \lambda^\theta \big(f(x) - \min_{\mathcal{H}} f\big).$$



Every nonempty sublevel set is star-shaped with respect to each minimizer, i.e., $x \in \operatorname{lev}_{\leq c} f \implies [\bar{x}, x] \subset \operatorname{lev}_{\leq c} f,\ \bar{x} \in \operatorname{argmin}_{\mathcal{H}} f$**.**

Minimizing properties

Proposition. Let $f : \mathcal{H} \to \mathbb{R}$ satisfy condition (C) and let $x : [0, +\infty[\to \mathcal{H}$ be a solution of (SD). Then, for every $\bar{x} \in \operatorname{argmin}_{\mathcal{H}} f$, the following assertions hold:

(i) Fejér monotonicity:

$$t \longmapsto \|x(t) - \bar{x}\|^2 \text{ is non-increasing;}$$

(ii) Objective-gap integrability:

$$\int_0^\infty \big(f(x(\tau)) - \min_{\mathcal{H}} f\big)\, \mathrm{d}\tau < +\infty.$$

Corollary. Under the above hypotheses, suppose that $f : \mathcal{H} \to \mathbb{R}$ is weakly (sequentially) lower semi-continuous.[1] Then there exists $\bar{x} \in \operatorname{argmin}_{\mathcal{H}} f$ such that

$$\text{w}-\lim_{t \to +\infty} x(t) = \bar{x}.$$

[1] A function $f : \mathcal{H} \to \mathbb{R}$ is weakly (sequentially) lower semi-continuous at $x \in \mathcal{H}$ if, for every sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ such that $x_n \rightharpoonup x$, it holds that

$$f(x) \leq \liminf_{n \to +\infty} f(x_n).$$

Asymptotic estimates

Theorem. Let $f : \mathcal{H} \to \mathbb{R}$ satisfy condition (C) and let $x : [0, +\infty[\to \mathcal{H}$ be a solution of (SD). Then the following assertions hold:

(i) Weighted "finite-energy property":

$$\int_0^\infty \tau \|\dot{x}(\tau)\|^2 \, \mathrm{d}\tau < +\infty;$$

(ii) Refined objective-value decay:

$$f(x(t)) - \min_{\mathcal{H}} f = o\left(\frac{1}{t}\right) \text{ as } t \to +\infty.$$

Lyapunov identity. Let $\rho : [0, +\infty[\to]0, +\infty[$ be continuously differentiable (to be chosen). For every $\bar{x} \in \operatorname{argmin}_{\mathcal{H}} f$, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|x(t) - \bar{x}\|^2 + \frac{1}{\rho(t)}\big(f(x(t)) - \min_{\mathcal{H}} f\big)\right)$$
$$- \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\rho(t)}\right)\big(f(x(t)) - \min_{\mathcal{H}} f\big) + \frac{1}{\rho(t)}\|\nabla f(x(t))\|^2$$
$$+ \langle \nabla f(x(t)), x(t) - \bar{x} \rangle = 0.$$

Lyapunov mechanism

Geometry of f and estimates	**Choice of ρ** $(\varepsilon > 0)$	**Rate for** $f(x(t)) - \min_{\mathcal{H}} f$
Condition (C) $\langle \nabla f(x), x - \bar{x} \rangle \geq \theta\big(f(x) - \min_{\mathcal{H}} f\big)$	$\rho(t) = \dfrac{1}{\theta(t+\varepsilon)}$	$o\left(\dfrac{1}{t}\right)$
Convexity $\langle \nabla f(x), x - \bar{x} \rangle \geq f(x) - \min_{\mathcal{H}} f$	$\rho(t) = \dfrac{1}{t+\varepsilon}$	$\mathcal{O}\left(\dfrac{1}{t}\right)$
μ**-strong convexity** $\langle \nabla f(x), x - \bar{x} \rangle \geq \mu \\|x - \bar{x}\\|^2$; $\\|\nabla f(x)\\|^2 \geq 2\mu\big(f(x) - \min_{\mathcal{H}} f\big)$	$\rho(t) \equiv 2\mu$	$\mathcal{O}\big(\mathrm{e}^{-2\mu t}\big)$

Little-o estimate. The weighted "finite-energy property" yields

$$t\big(f(x(t)) - \min_{\mathcal{H}} f\big) = \int_t^{\infty} t\|\dot{x}(\tau)\|^2 \,\mathrm{d}\tau \leq \int_t^{\infty} \tau\|\dot{x}(\tau)\|^2 \,\mathrm{d}\tau \to 0.$$

Quadratic growth

Quadratic growth condition. Suppose that there exists $\alpha > 0$ such that, for every $x \in \mathcal{H}$ and $\bar{x} \in \operatorname{argmin}_{\mathcal{H}} f$,

$$f(x) - \min_{\mathcal{H}} f \geq \alpha \|x - \bar{x}\|^2. \tag{QG}$$

Proposition. Let $f : \mathcal{H} \to \mathbb{R}$ satisfy condition (C) with $\theta > 0$ and condition (QG) with $\alpha > 0$, and let $x : [0, +\infty[\to \mathcal{H}$ be a solution of (SD). Then, for $\bar{x} \in \operatorname{argmin}_{\mathcal{H}} f$, it holds that

$$\begin{aligned} f(x(t)) - \min_{\mathcal{H}} f &= \mathcal{O}\big(\mathrm{e}^{-2\alpha\theta t}\big) \text{ as } t \to +\infty; \\ \|x(t) - \bar{x}\|^2 &= \mathcal{O}\big(\mathrm{e}^{-2\alpha\theta t}\big) \text{ as } t \to +\infty; \\ \|\dot{x}(t)\|^2 &= \mathcal{O}\big(\mathrm{e}^{-2\alpha\theta t}\big) \text{ as } t \to +\infty. \end{aligned}$$

Remark. (C) + (QG) $\implies$ Polyak–Łojasiewicz (PL) inequality

$$\exists \mu > 0 \; \forall x \in \mathcal{H} \quad \|\nabla f(x)\|^2 \geq 2\mu\big(f(x) - \min_{\mathcal{H}} f\big)$$

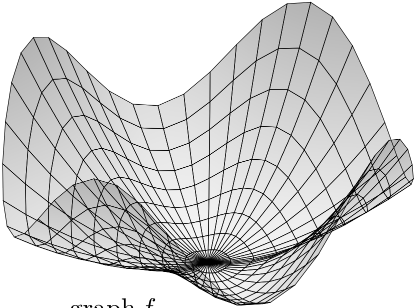
with $\mu = \alpha\theta^2/2$.

Numerical experiment I

Example. Let $\mathcal{H} = \mathbb{R}^2$ and consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

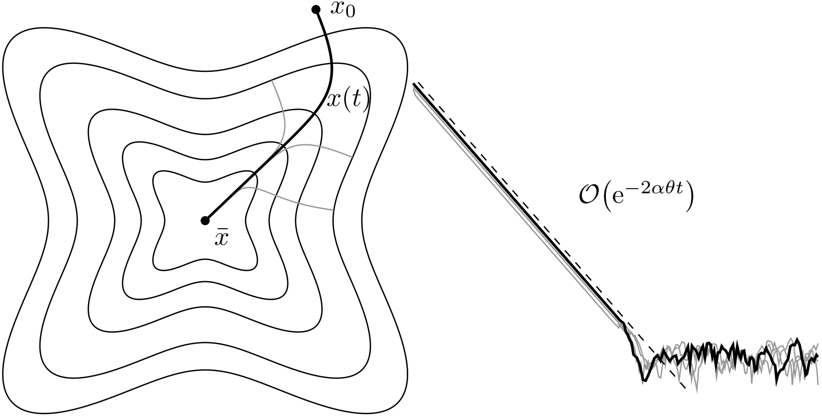
$$f(x) = \begin{cases} \phi(x)\|x\|^2, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

where $\phi \in \mathcal{C}^1(\mathbb{R}^2 \setminus \{0\}; \mathbb{R})$ takes the form $\phi(x) = (1/2 + \varepsilon \cos(k \arg x))$ with $\varepsilon = 1/4$ and $k = 4$.



Numerical experiment I (cont'd)

Illustration. (C) with $\theta = 2$ and (QG) with $\alpha = 1/2 - \varepsilon$



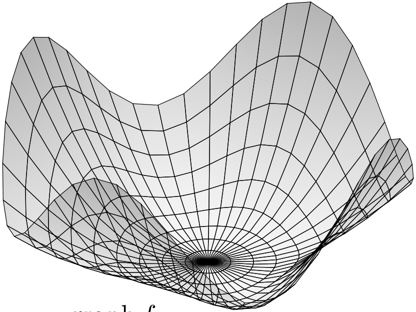
$\{x(t) : t \geq 0\}$ $\qquad$ $\|x(t) - \bar{x}\|^2/2$

Numerical experiment II

Example. Let $\mathcal{H} = \mathbb{R}^2$ and consider now $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \phi(x)\|x\|^2 \, \mathrm{e}^{-1/\|x\|^2}, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

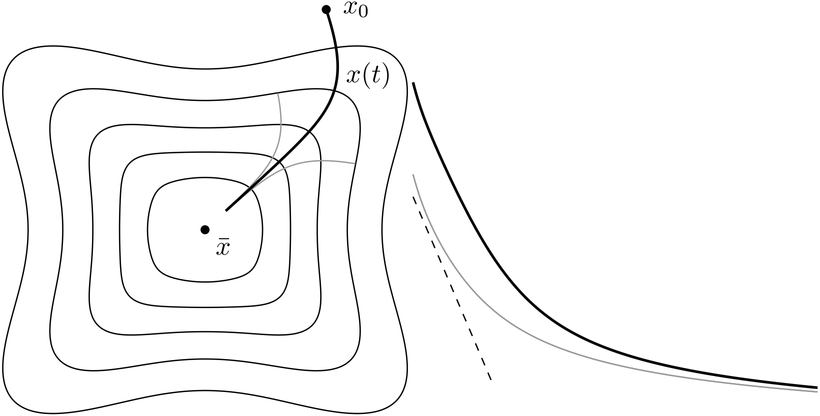
where $\phi \in \mathcal{C}^1(\mathbb{R}^2 \setminus \{0\}; \mathbb{R})$ takes the form $\phi(x) = (1/2 + \varepsilon \cos(k \arg x))$ with $\varepsilon = 1/4$ and $k = 4$.



graph f

Numerical experiment II (cont'd)

Illustration. (C) with $\theta = 2$ and (QG) not satisfied



$\{x(t) : t \geq 0\}$ $\quad$ $\|x(t) - \bar{x}\|^2/2$

Conclusions

Continuous steepest descent

$$\dot{x} + \nabla f(x) = 0$$

(C)

Refined asymptotic estimates

$$\int_0^{\infty} \tau \|\dot{x}(\tau)\|^2 \, \mathrm{d}\tau < +\infty;$$

$$f(x(t)) - \min_{\mathcal{H}} f = o\left(\frac{1}{t}\right)$$

(QG)

(C)$\times$(QG)

Exponential convergence

$$f(x(t)) - \min_{\mathcal{H}} f = \mathcal{O}\left(\mathrm{e}^{-2\alpha\theta t}\right)$$

Classical continuous steepest descent retains the convex-case asymptotics under (C)**; adding** (QG) **yields exponential convergence.**

Thank you for your attention!

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